Review of Math Essentials

Appendix A

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Appendix A: Review of Math Essentials

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A.1 Summation

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

- The symbol Σ is the capital Greek letter sigma, and means "the sum of."
- The letter *i* is called the **index of summation**. This letter is arbitrary and may also appear as *t*, *j*, or *k*.
- The expression $\sum_{i=1}^{n} x_i$ is read "the sum of the terms x_i, from *i* equal one to *n*."
- The numbers 1 and *n* are the **lower limit** and **upper limit** of summation.

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A.1 Summation

Rules of summation operation:

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

$$\sum_{i=1}^{n} a x_i = a \sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} a = a + a + \cdots + a = na$$

$$\sum_{i=1}^{n} (x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$$

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A.1 Summation

$$\sum_{i=1}^{n} (a x_i + b y_i) = a \sum_{i=1}^{n} x_i + b \sum_{i=1}^{n} y_i$$

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

$$\sum_{i=1}^{n} (x_i - \bar{x}) = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \bar{x} = \sum_{i=1}^{n} x_i - n\bar{x} = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i = 0$$

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A.1 Summation

$$\sum_{i=1}^{n} f(x_i) = f(x_1) + f(x_2) + \cdots + f(x_n)$$

$$= \sum_{i} f(x_i) \quad \text{("Sum over all values of the index } i\text{")}$$

$$= \sum_{x} f(x) \quad \text{("Sum over all possible values of } X\text{")}$$

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A.1 Summation

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j) = \sum_{i=1}^{m}\sum_{j=1}^{n}(x_i + y_j)$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j) = \sum_{i=1}^{m}[f(x_i, y_1) + f(x_i, y_2) + \cdots + f(x_i, y_n)]$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j) = \sum_{j=1}^{n}\sum_{i=1}^{m} f(x_i, y_j)$$

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A.2 Some Basics

- **A.2.1 Numbers**

Integers are the whole numbers, 0, 1, 2, 3, …. .

Rational numbers can be written as *a*/*b*, where *a* and *b* are integers, and $b \neq 0$.

The **real numbers** can be represented by points on a line. There are an uncountable number of real numbers and they are not all rational. Numbers such as $\pi \cong 3.1415927$ and $\sqrt{2}$ are said to be **irrational** since they cannot be expressed as ratios, and have only decimal representations. Numbers like $\sqrt{-2}$ are not real numbers.

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A.2 Some Basics

The **absolute value** of a number is denoted $|a|$. It is the positive part of the number, so that $|3| = 3$ and $|-3| = 3$.

Basic rules about **Inequalities:**

If $a < b$, then $a + c < b + c$

If $a < b$, then $\begin{cases} ac < bc & \text{if } c > 0 \\ ac > bc & \text{if } c < 0 \end{cases}$

If $a < b$ and $b < c$, then $a < c$

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A.2 Some Basics

- **A.2.2 Exponents**

$x^n = xx\cdots x$ (n terms) if n is a positive integer

$x^0 = 1$ if $x \neq 0$. 0^0 is not defined

Rules for working with exponents, assuming x and y are real, m and n are integers, and a and b are rational:

$$x^{-n} = \frac{1}{x^n} \text{ if } x \neq 0$$

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A.2 Some Basics

$$x^{1/n} = \sqrt[n]{x}$$

$$x^{m/n} = \left(x^{1/n}\right)^m$$

$$x^{m/n} = \left(x^{1/n}\right)^m$$

$$x^a x^b = x^{a+b}$$

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$$

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A.2.3 Scientific Notation

$$510,000 \times .00000034 = (5.1\times 10^5)\times(3.4\times 10^{-7})$$

$$= (5.1\times 3.4)\times(10^5 \times 10^{-7})$$

$$= 17.34\times 10^{-2}$$

$$= .1734$$

$$\frac{510,000}{.00000034} = \frac{5.1\times 10^5}{3.4\times 10^{-7}} = \frac{5.1}{3.4}\times\frac{10^5}{10^{-7}} = 1.5\times 10^{12}$$

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A.2.4 Logarithms and the number e

$$\ln(x) = \ln(e^b) = b$$

Rules:

$$\ln(xy) = \ln(x) + \ln(y)$$
$$\ln(x/y) = \ln(x) - \ln(y)$$
$$\ln(x^a) = a\ln(x)$$

A.2.4 Logarithms and the number e

Table A.1 **Some Natural Logarithms**

x	$\ln(x)$
1	0
10	2.3025851
100	4.6051702
1000	6.9077553
10,000	9.2103404
100,000	11.512925
1,000,000	13.815511

A.2.4 Logarithms and the number e

$$x = e^{\ln(x)} = \exp\left[\ln(x)\right]$$

The exponential function is the **antilogarithm** because we can recover the value of x using it.

A.3 Linear Relationships

$$y = \beta_1 + \beta_2 x \quad \text{(A.1)}$$

$$\Delta y = \beta_2 \Delta x \quad \text{(A.2)}$$

$$y = \beta_1 + \beta_2 x = \beta_1 + \beta_2 0 = \beta_1$$

A.3 Linear Relationships

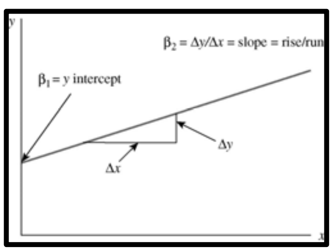

Figure A.1 A linear relationship

A.3 Linear Relationships

$$\frac{dy}{dx} = \beta_2 \quad \text{(A.3)}$$

$$y = \beta_1 + \beta_2 x_2 + \beta_3 x_3 \quad \text{(A.4)}$$

$$\beta_2 = \frac{\Delta y}{\Delta x_2} \text{ given that } x_3 \text{ is held constant}$$
$$\beta_3 = \frac{\Delta y}{\Delta x_3} \text{ given that } x_2 \text{ is held constant} \quad \text{(A.5)}$$

A.3 Linear Relationships

$$Q = \beta_1 + \beta_2 L + \beta_3 K$$

$$\beta_2 = \frac{\Delta Q}{\Delta L} \text{ given that capital } K \text{ is held constant} \quad \text{(A.6)}$$
$$= MP_L \text{, the marginal product of labor input}$$

$$\frac{\partial y}{\partial x_2} = \beta_2, \quad \frac{\partial y}{\partial x_3} = \beta_3$$

A.3.1 Elasticity

$$\varepsilon_{yx} = \frac{\Delta y / y}{\Delta x / x} = \frac{\Delta y}{\Delta x} \times \frac{x}{y} = slope \times \frac{x}{y} \quad \text{(A.7)}$$

$\Delta y/y$ = the relative change in y, which is a decimal (A.8a)

$\%\Delta y$ = percentage change in y (A.8b)

$\%\Delta y = 100 \times \frac{\Delta y}{y}$ (A.8c)

A.4 Nonlinear Relationships

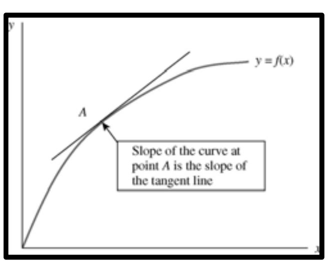

Figure A.2 A nonlinear relationship

A.4 Nonlinear Relationships

$$dy = \beta_2 dx \qquad \text{(A.9)}$$

$$\varepsilon_{yx} = \frac{dy/y}{dx/x} = \frac{dy}{dx} \times \frac{x}{y} = slope \times \frac{x}{y} \qquad \text{(A.10)}$$

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A.4 Nonlinear Relationships

Table A.2 **Some Useful Functions, Their Derivatives and Elasticities**

Name	Function	Slope = dy/dx	Elasticity
Linear	$y = \beta_1 + \beta_2 x$	β_2	$\beta_2 \frac{x}{y}$
Quadratic	$y = \beta_1 + \beta_2 x + \beta_3 x^2$	$\beta_2 + 2\beta_3 x$	$(\beta_2 + 2\beta_3 x)\frac{x}{y}$
Cubic	$y = \beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^3$	$\beta_2 + 2\beta_3 x + 3\beta_4 x^2$	$(\beta_2 + 2\beta_3 x + 3\beta_4 x^2)\frac{x}{y}$
Reciprocal	$y = \beta_1 + \beta_2 \frac{1}{x}$	$-\beta_2 \frac{1}{x^2}$	$-\beta_2 \frac{1}{xy}$
Log-log	$\ln(y) = \beta_1 + \beta_2 \ln(x)$	$\beta_2 \frac{y}{x}$	β_2
Log-linear	$\ln(y) = \beta_1 + \beta_2 x$	$\beta_2 y$	$\beta_2 x$
Linear-log	$y = \beta_1 + \beta_2 \ln(x)$	$\beta_2 \frac{1}{x}$	$\beta_2 \frac{1}{y}$

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A.4 Nonlinear Relationships

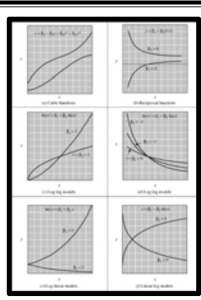

Figure A.3 Alternative Functional Forms

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A.4.1 Quadratic Function

$$y = \beta_1 + \beta_2 x + \beta_3 x^2$$

If $\beta_3 > 0$, then the curve is U-shaped, and representative of average or marginal cost functions, with increasing marginal effects. If $\beta_3 < 0$, then the curve is an inverted-U shape, useful for total product curves, total revenue curves, and curves that exhibit diminishing marginal effects.

$$dy/dx = \beta_2 + 2\beta_3 x = 0, \text{ or } x = -\beta_2 / (2\beta_3)$$

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A.4.2 Cubic Function

$$y = \beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^3$$

- Cubic functions can have two **inflection points**, where the function crosses its tangent line, and changes from concave to convex, or vice versa.
- Cubic functions can be used for total cost and total product curves in economics. The derivative of total cost is marginal cost, and the derivative of total product is marginal product.
- If the “total” curves are cubic then the “marginal” curves are quadratic functions, a U-shaped curve for marginal cost, and an inverted-U shape for marginal product.

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A.4.3 Reciprocal Function

$$y = \beta_1 + \beta_2 \frac{1}{x} = \beta_1 + \beta_2 x^{-1}$$

- Example: the Phillips Curve

$$\%\Delta w_t = \frac{w_t - w_{t-1}}{w_{t-1}} \times 100$$

$$\%\Delta w_t = \beta_1 + \beta_2 \frac{1}{u_t}$$

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A.4.4 Log-Log Function

$$\ln(y) = \beta_1 + \beta_2 \ln(x)$$

- In order to use this model all values of y and x must be positive. The slopes of these curves change at every point, but the elasticity is constant and equal to β_2.

A.4.5 Log-Linear Function

$$\ln(y) = \beta_1 + \beta_2 x$$

- Both its slope and elasticity change at each point and are the same sign as β_2.

$$\exp\left[\ln(y)\right] = y = \exp(\beta_1 + \beta x)$$

- The slope at any point is $\beta_2 y$, which for $\beta_2 > 0$ means that the marginal effect increases for larger values of y.

A.4.6 Approximating Logarithms

$$\ln\left(y_1\right) \cong \ln(y_0) + \frac{1}{y_0}(y_1 - y_0) \qquad \text{(A.11)}$$

$$\ln(1 + x) \cong x$$

$$\ln\left(y_1\right) - \ln(y_0) = \Delta \ln\left(y\right) \cong \frac{1}{y_0}(y_1 - y_0) = \frac{\Delta y}{y_0} = \text{relative change in } y$$

A.4.6 Approximating Logarithms

$$
\begin{aligned}
100\Delta\ln(y) &= 100\left[\ln(y_1) - \ln(y_0)\right] \\
&\cong 100 \times \frac{\Delta y}{y_0} \\
&= \%\Delta y = \text{ percentage change in } y
\end{aligned}
\qquad \text{(A.12)}
$$

A.4.6 Approximating Logarithms

$$\%\Delta y = 100 \times \frac{\Delta y}{y_0} = 100(y_1 - 1)$$

$$\%\text{ approximation error} = 100\left[\frac{\%\Delta y - 100\Delta\ln(y)}{100\Delta\ln(y)}\right] = 100\left[\frac{(y_1 - 1) - \ln(y_1)}{\ln(y_1)}\right]$$

A.4.6 Approximating Logarithms

Table A.3 **Log Difference Approximation Error**

y_1	$\%\Delta y$	$100\Delta\ln(y)(\%)$	Approximation error (%)
1.01	1.00	0.995	0.50
1.05	5.00	4.88	2.48
1.10	10.00	9.53	4.92
1.15	15.00	13.98	7.33
1.20	20.00	18.23	9.70
1.25	25.00	22.31	12.04

A.4.7 Approximating Logarithms in the Log-Linear Model

$$100\left[\ln(y_1) - \ln(y_0)\right] \cong \%\Delta y$$
$$= 100\beta_2\left(x_1 - x_0\right) \quad \text{(A.13)}$$
$$= (100\beta_2) \times \Delta x$$

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A.4.8 Linear-Log Function

$$y_0 = \beta_1 + \beta_2 \ln(x_0)$$
$$y_1 = \beta_1 + \beta_2 \ln(x_1)$$

$$\Delta y = y_1 - y_0 = \beta_2\left[\ln(x_1) - \ln(x_0)\right]$$
$$= \frac{\beta_2}{100} \times 100\left[\ln(x_1) - \ln(x_0)\right]$$
$$\cong \frac{\beta_2}{100}\left(\%\Delta x\right)$$

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A.4.8 Linear-Log Function

$$y = \beta_1 + \beta_2 \ln(x) = 0 + 500\ln(x)$$

$$\Delta y = \frac{\beta_2}{100}\left(\%\Delta x\right) = \frac{500}{100} \times 10 = 50$$

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A.4.8 Linear-Log Function

$$y = \beta_1 + \beta_2 \ln(x) = 0 + 500\ln(x)$$

$$\Delta y = \frac{\beta_2}{100}(\%\Delta x) = \frac{500}{100} \times 10 = 50$$

Keywords

- absolute value
- antilogarithm
- asymptote
- ceteris paribus
- cubic function
- derivative
- double summation
- *e*
- elasticity
- exponential function
- exponents
- inequalities
- integers
- intercept
- irrational numbers
- linear relationship
- logarithm
- log-linear function
- log-log function
- marginal effect
- natural logarithm
- nonlinear relationship
- partial derivative
- percentage change
- Phillips curve
- quadratic function
- rational numbers
- real numbers
- reciprocal function
- relative change
- scientific notation
- slope
- summation sign