Moment of Inertia of a Composite Area

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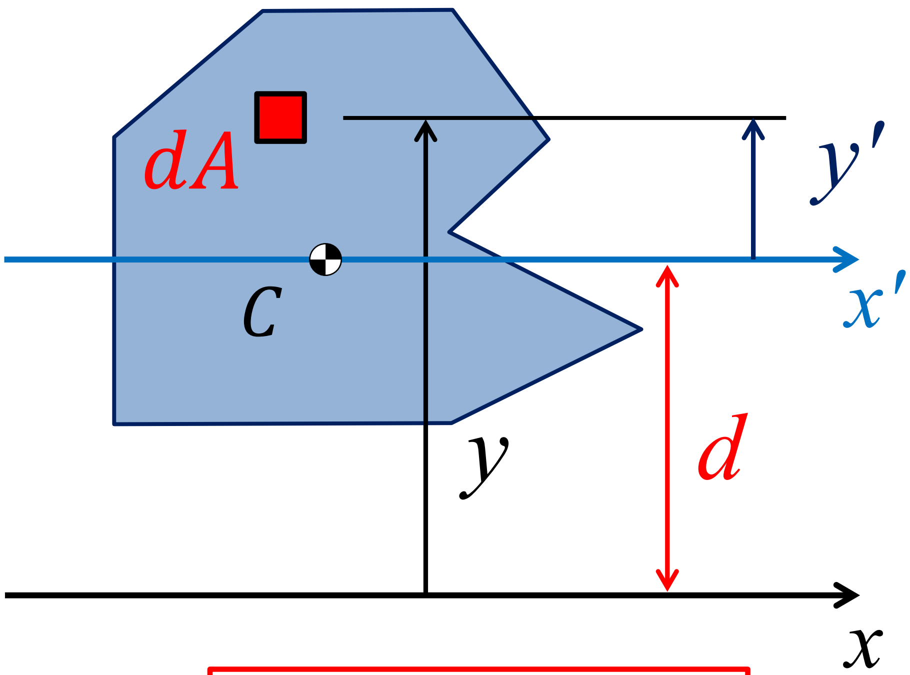
Recall the Parallel Axis Theorem



$$I_x = \bar{I}_{x'} + d^2 A$$

General Form

$$I = \bar{I} + d^2 A$$

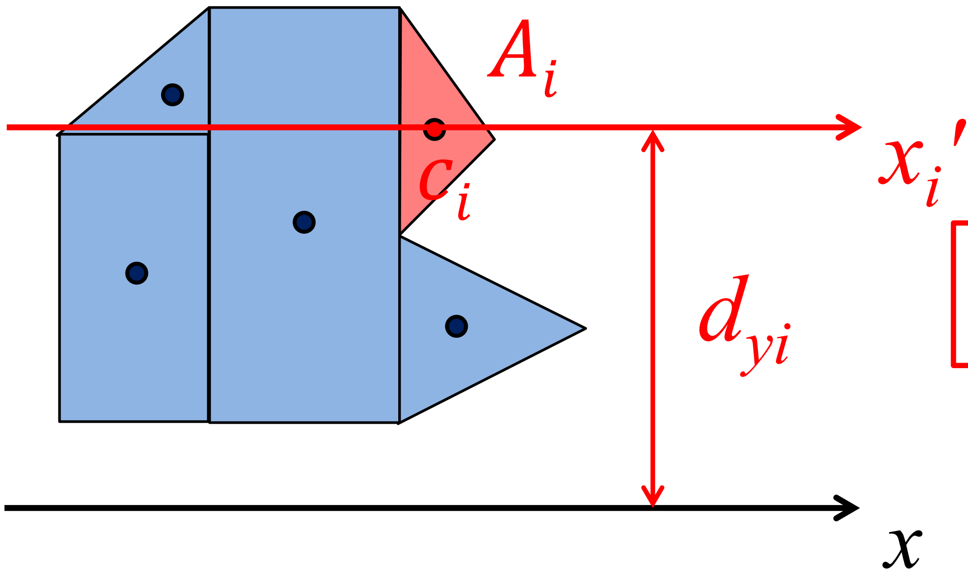
Centroidal Moment of Inertia

$$\bar{I}_{x'} = \iint y'^2 \, dA$$

Parallel Axis Theorem

If we know the moment of inertia of a body about an axis passing through its centroid, we can calculate the body's moment of inertia about any parallel axis

If We Can Divide an Area into Simple Shapes With Known Centroid

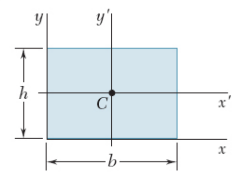
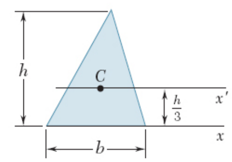
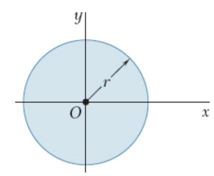
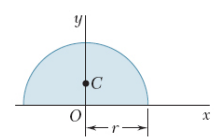
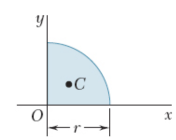
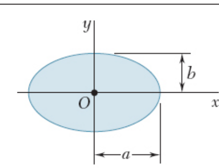


$$I_{xi} = \bar{I}_{xi'} + d_{yi}^2 A_i$$

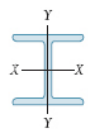
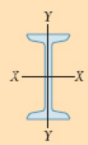
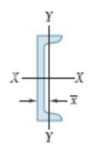
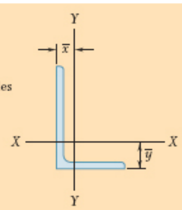
Moment of Inertia of the entire area about the *x* axis

$$I_x = \sum I_{xi} = \sum \bar{I}_{xi'} + \sum d_{yi}^2 A_i$$

Tabulated Centroidal Moments of Inertia Can be Found in the Textbook

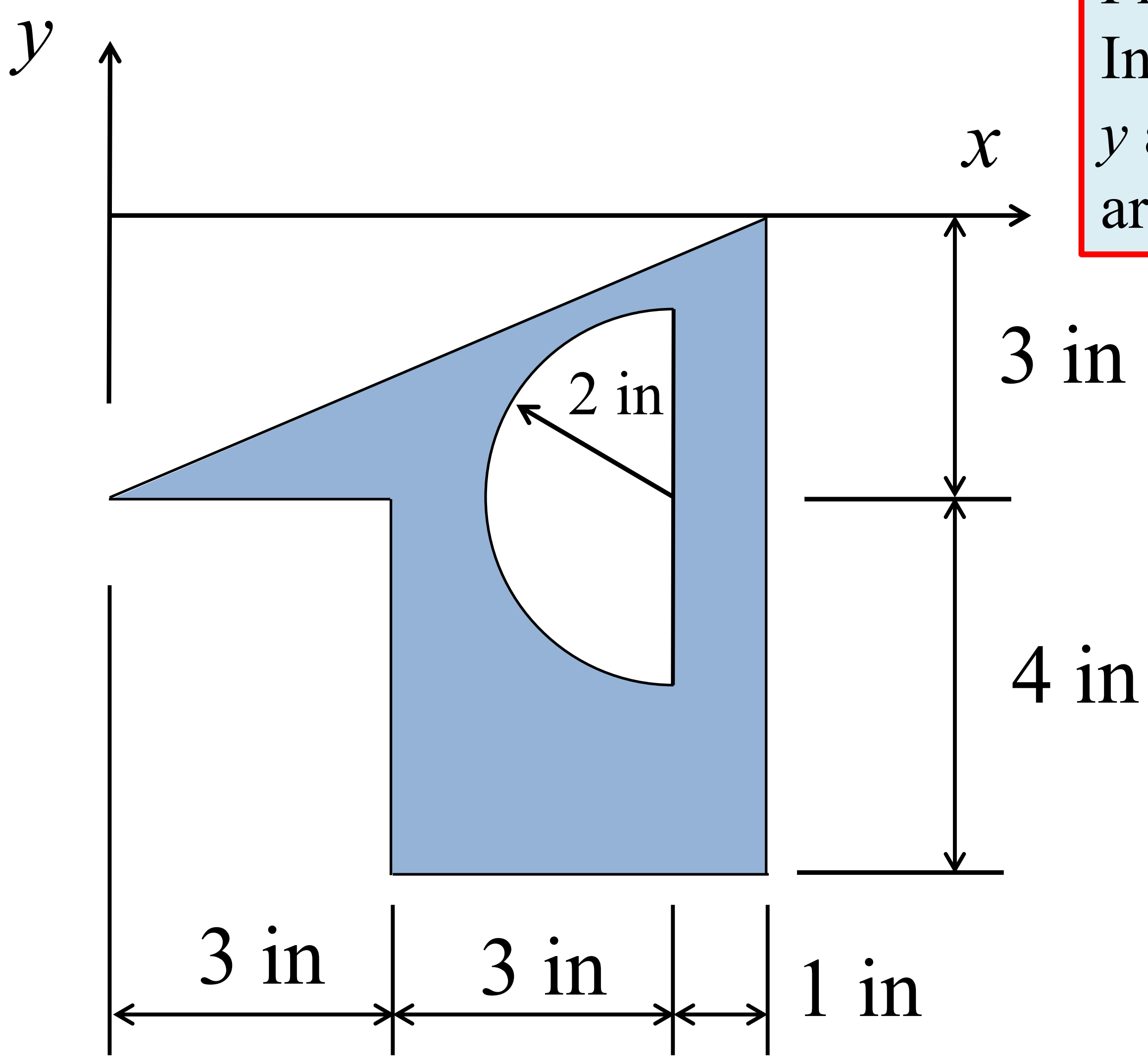
Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$
Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$
Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Quarter circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Ellipse		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

Tabulated Centroidal Moments of Inertia Can be Found in the Textbook

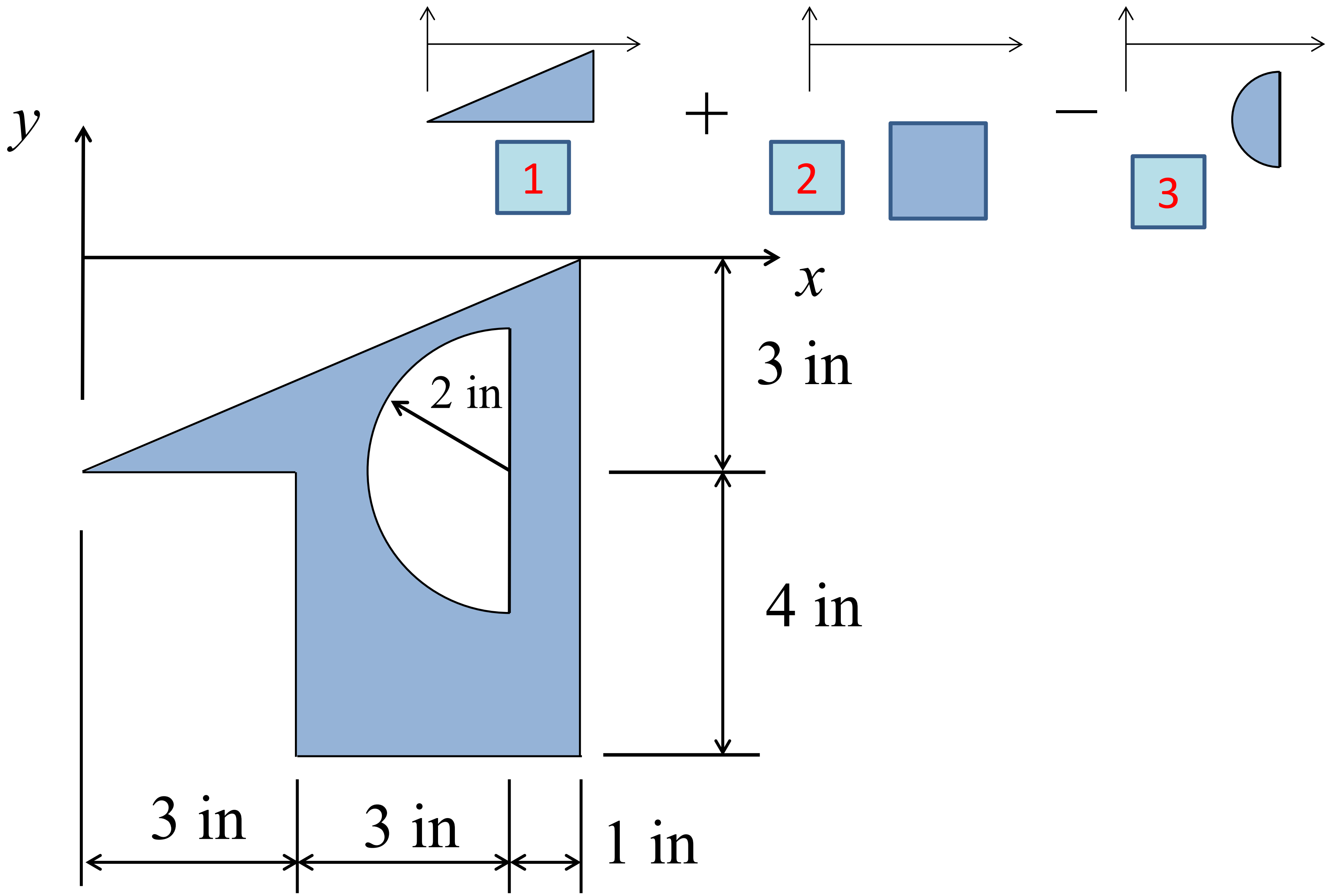
	Designation	Area in²	Depth in.	Width in.	Axis X–X: $\bar{I}_x$, in⁴	Axis X–X: $\bar{k}_x$, in.	Axis X–X: $\bar{y}$, in.	Axis Y–Y: $\bar{I}_y$, in⁴	Axis Y–Y: $\bar{k}_y$, in.	Axis Y–Y: $\bar{x}$, in.
W Shapes (Wide-Flange Shapes) 	W18 × 76†	22.3	18.2	11.0	1330	7.73		152	2.61	
	W16 × 57	16.8	16.4	7.12	758	6.72		43.1	1.60	
	W14 × 38	11.2	14.1	6.77	385	5.87		26.7	1.55	
	W8 × 31	9.12	8.00	8.00	110	3.47		37.1	2.02	
S Shapes (American Standard Shapes) 	S18 × 54.7†	16.0	18.0	6.00	801	7.07		20.7	1.14	
	S12 × 31.8	9.31	12.0	5.00	217	4.83		9.33	1.00	
	S10 × 25.4	7.45	10.0	4.66	123	4.07		6.73	0.950	
	S6 × 12.5	3.66	6.00	3.33	22.0	2.45		1.80	0.702	
C Shapes (American Standard Channels) 	C12 × 20.7†	6.08	12.0	2.94	129	4.61		3.86	0.797	0.698
	C10 × 15.3	4.48	10.0	2.60	67.3	3.87		2.27	0.711	0.634
	C8 × 11.5	3.37	8.00	2.26	32.5	3.11		1.31	0.623	0.572
	C6 × 8.2	2.39	6.00	1.92	13.1	2.34		0.687	0.536	0.512
Angles 	L6 × 6 × 1‡	11.0			35.4	1.79	1.86	35.4	1.79	1.86
	L4 × 4 × $\frac{1}{2}$	3.75			5.52	1.21	1.18	5.52	1.21	1.18
	L3 × 3 × $\frac{1}{4}$	1.44			1.23	0.926	0.836	1.23	0.926	0.836
	L6 × 4 × $\frac{1}{2}$	4.75			17.3	1.91	1.98	6.22	1.14	0.981
	L5 × 3 × $\frac{1}{2}$	3.75			9.43	1.58	1.74	2.55	0.824	0.746
	L3 × 2 × $\frac{1}{4}$	1.19			1.09	0.953	0.980	0.390	0.569	0.487

Example Problem

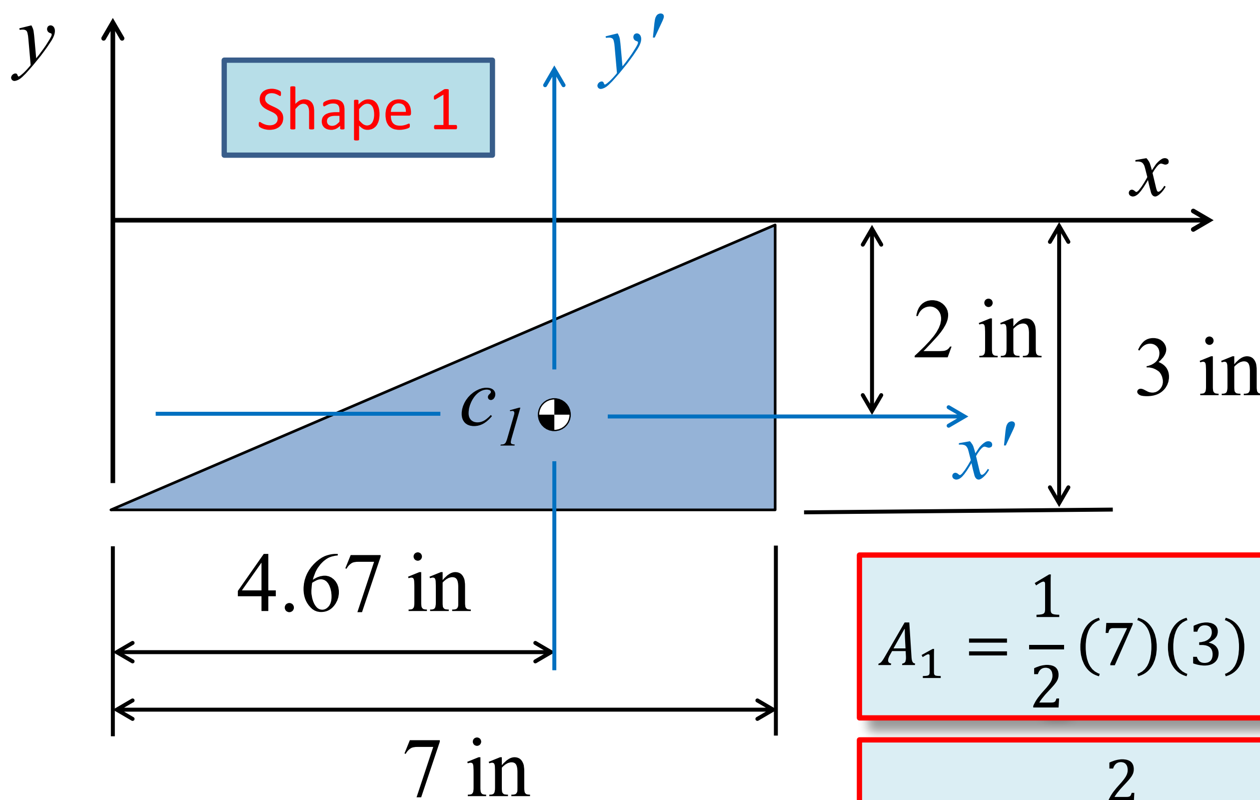
Find the Moment of Inertia about the *x* and *y* axis of the shaded area.



Divide Area into Simple Composite Shapes



Find the Area, Location of Centroid, and the Centroidal Moment of Inertia of Each Shape



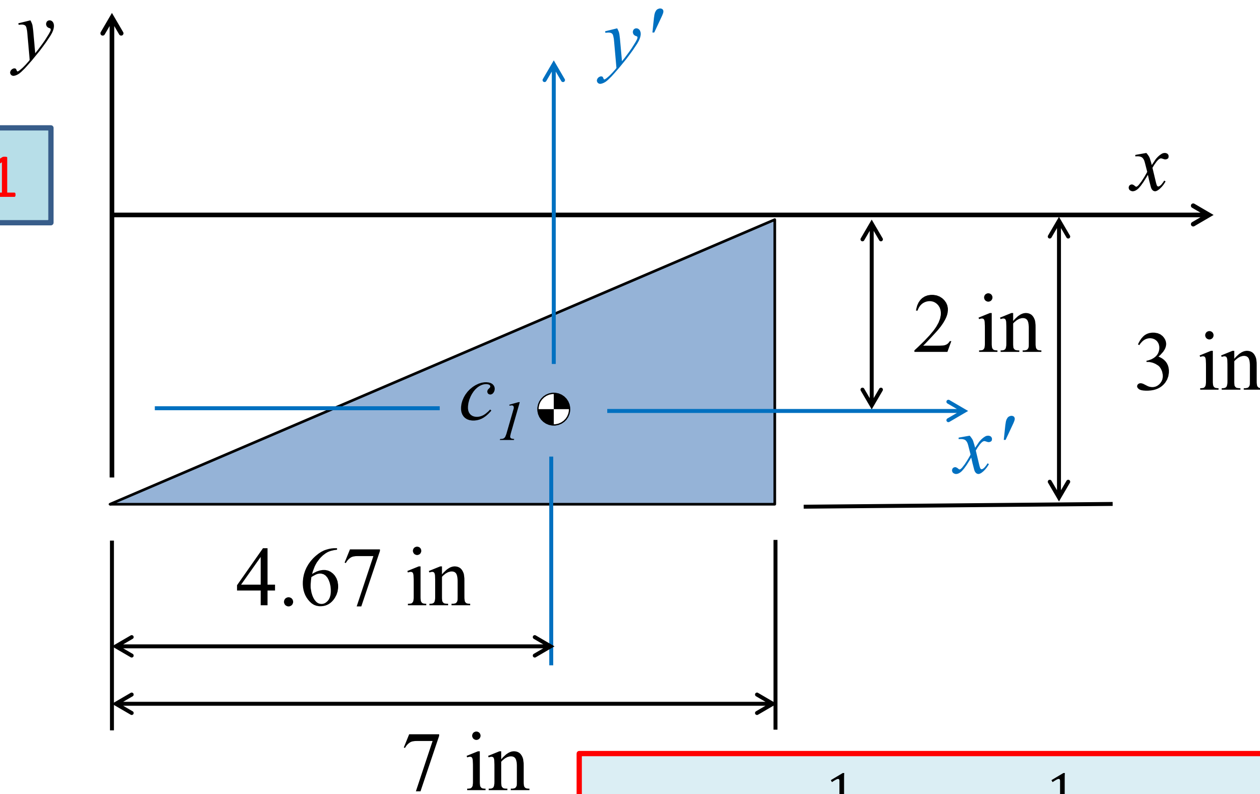
$$A_1 = \frac{1}{2}(7)(3) = 10.5\ \text{in}^2$$

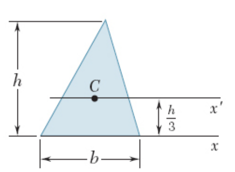
$$d_{x1} = \frac{2}{3}(7) = 4.67\ \text{in}$$

$$d_{y1} = -\frac{2}{3}(3) = -2.0\ \text{in}$$

Find the Area, Location of Centroid, and the Centroidal Moment of Inertia of Each Shape

Shape 1





Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$

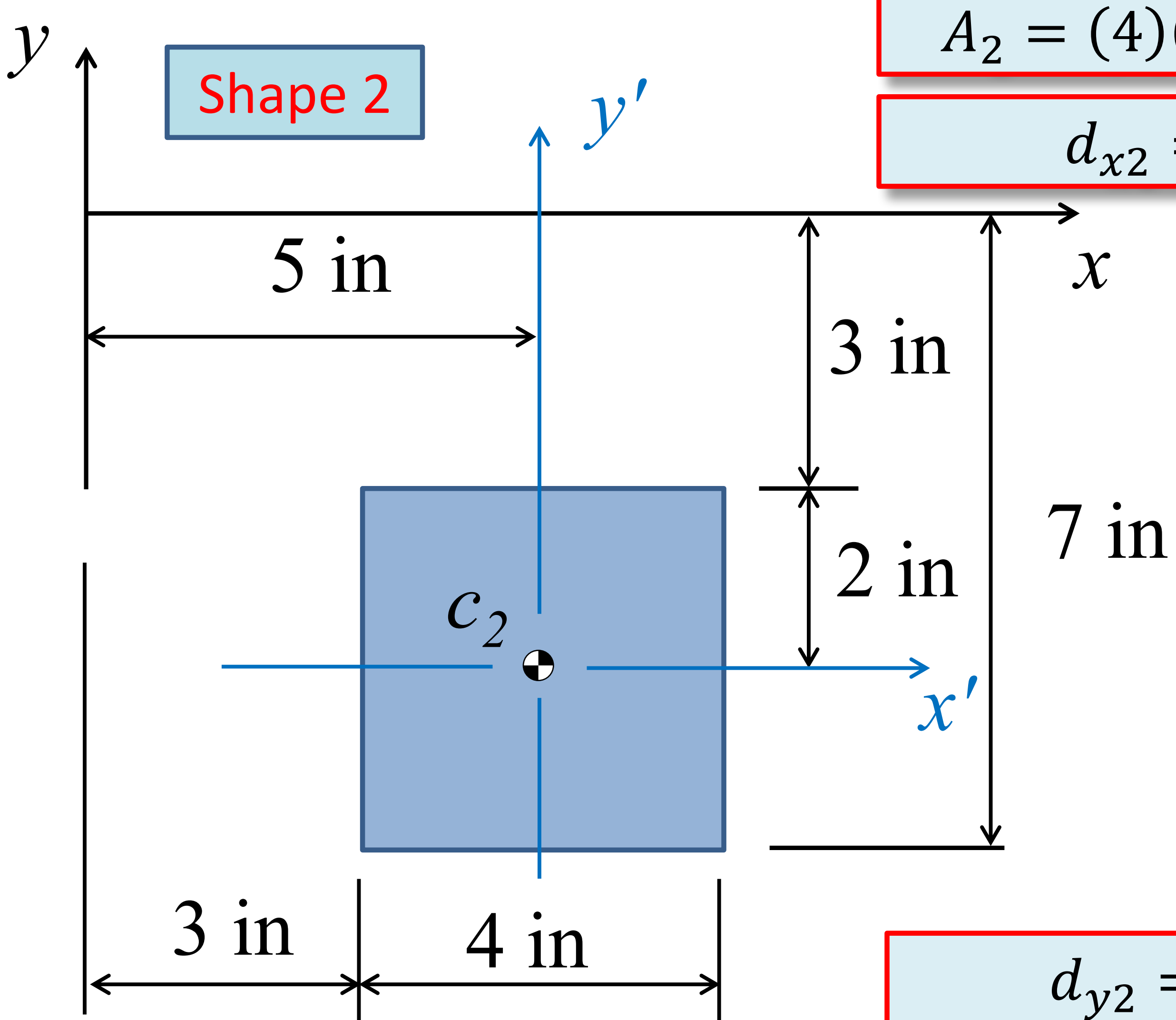
$$\bar{I}_{x1'} = \frac{1}{36}bh^3 = \frac{1}{36}(7)(3^3) = 5.25\ \text{in}^4$$

$$\bar{I}_{y1'} = \frac{1}{36}hb^3 = \frac{1}{36}(3)(7^3) = 28.5833\ \text{in}^4$$

Find Area and Location of Centroid of Each Shape Relative to Reference Coordinate Axes

$A_2 = (4)(4) = 16\ \text{in}^2$

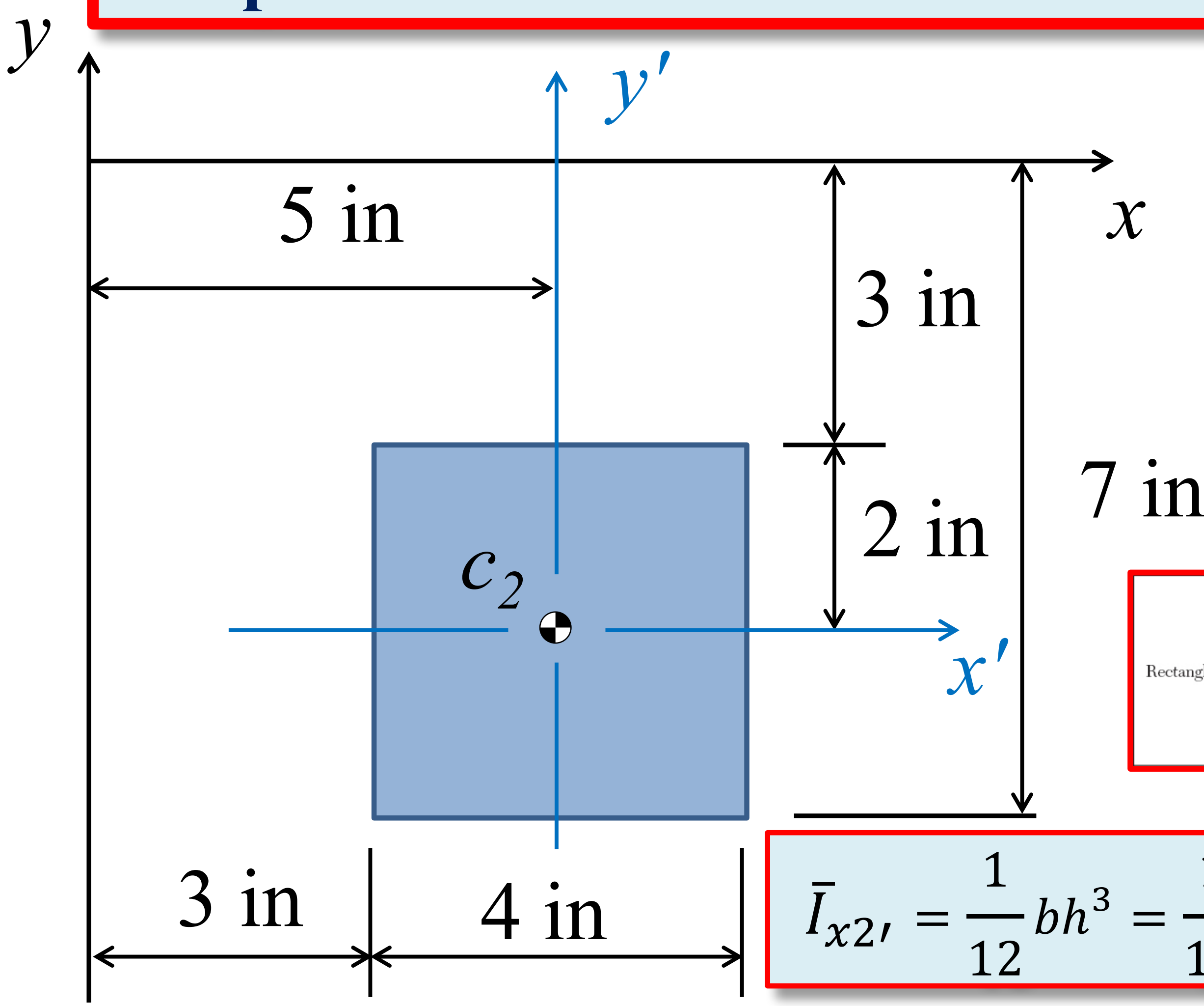
$d_{x2} = 5.0\ \text{in}$

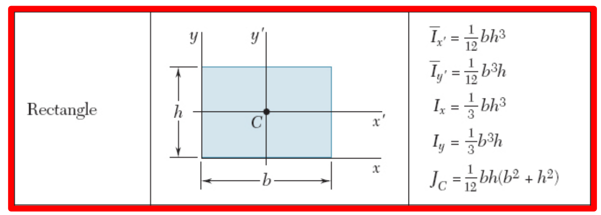


$d_{y2} = -5.0\ \text{in}$

Find Area and Location of Centroid of Each Shape Relative to Reference Coordinate Axes

Shape 2





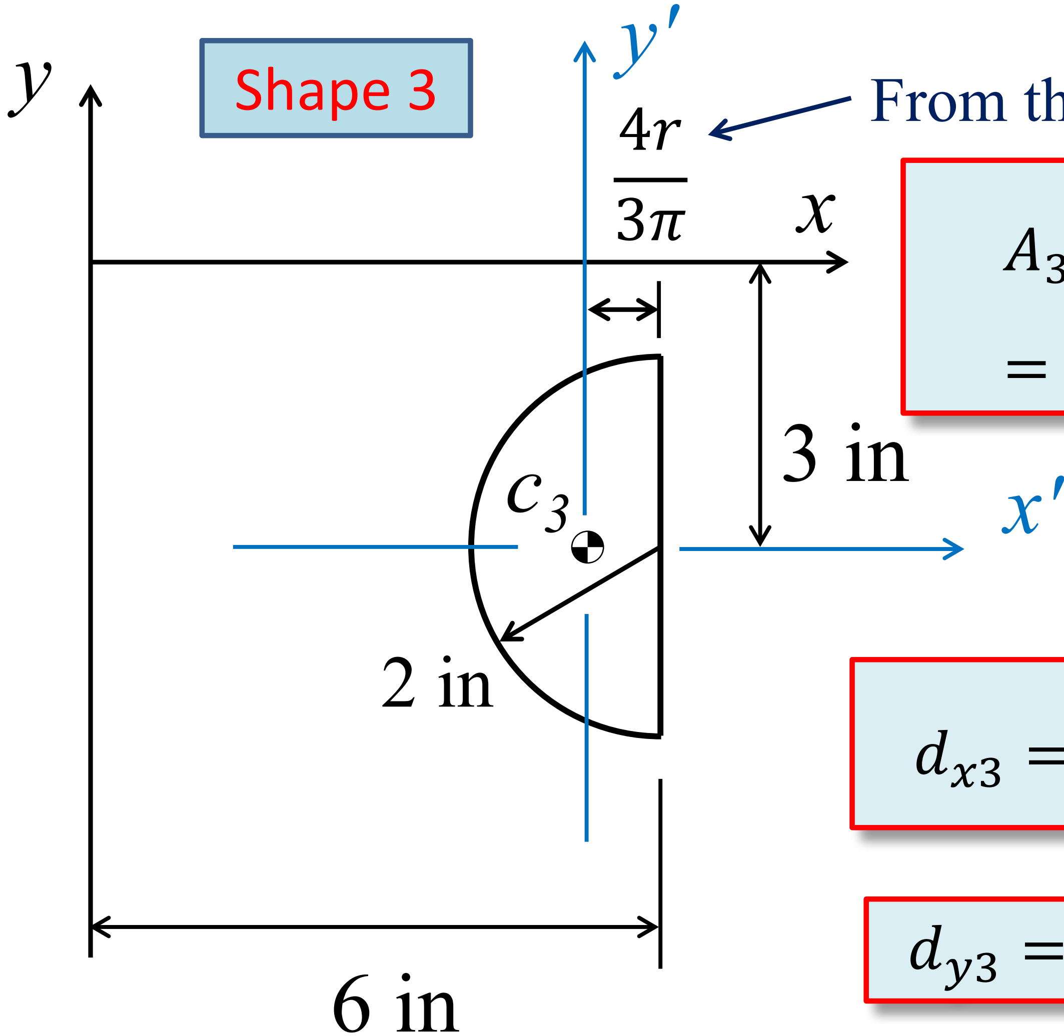
$$\bar{I}_{x'} = \frac{1}{12}bh^3$$
$$\bar{I}_{y'} = \frac{1}{12}b^3h$$
$$I_x = \frac{1}{3}bh^3$$
$$I_y = \frac{1}{3}b^3h$$
$$J_C = \frac{1}{12}bh(b^2 + h^2)$$

$$\bar{I}_{x2'} = \frac{1}{12}bh^3 = \frac{1}{12}(4)(4^3) = 21.333 \text{ in}^4$$

$$\bar{I}_{y2'} = \frac{1}{12}hb^3 = \frac{1}{12}(4)(4^3) = 21.333 \text{ in}^4$$

Find Area and Location of Centroid of Each Shape Relative to Reference Coordinate Axes

Shape 3



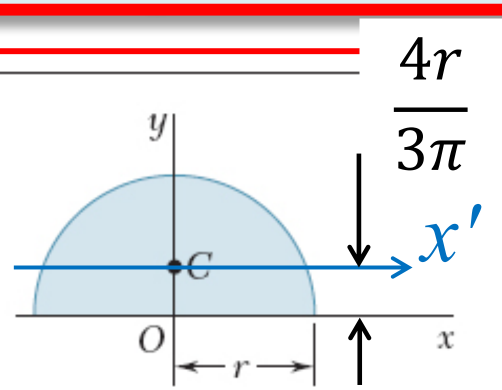
From the table

$$A_3 = -\frac{1}{2}\pi r^2 = -\frac{1}{2}\pi(2^2)$$
$$= -6.2832 \text{ in}^2$$

$$d_{x3} = 6 - \frac{4(2)}{3\pi} = 5.1512 \text{ in}$$

$$d_{y3} = -3 \text{ in}$$

Use Parallel Axis Theorem to Complete Table for Semicircle

Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$

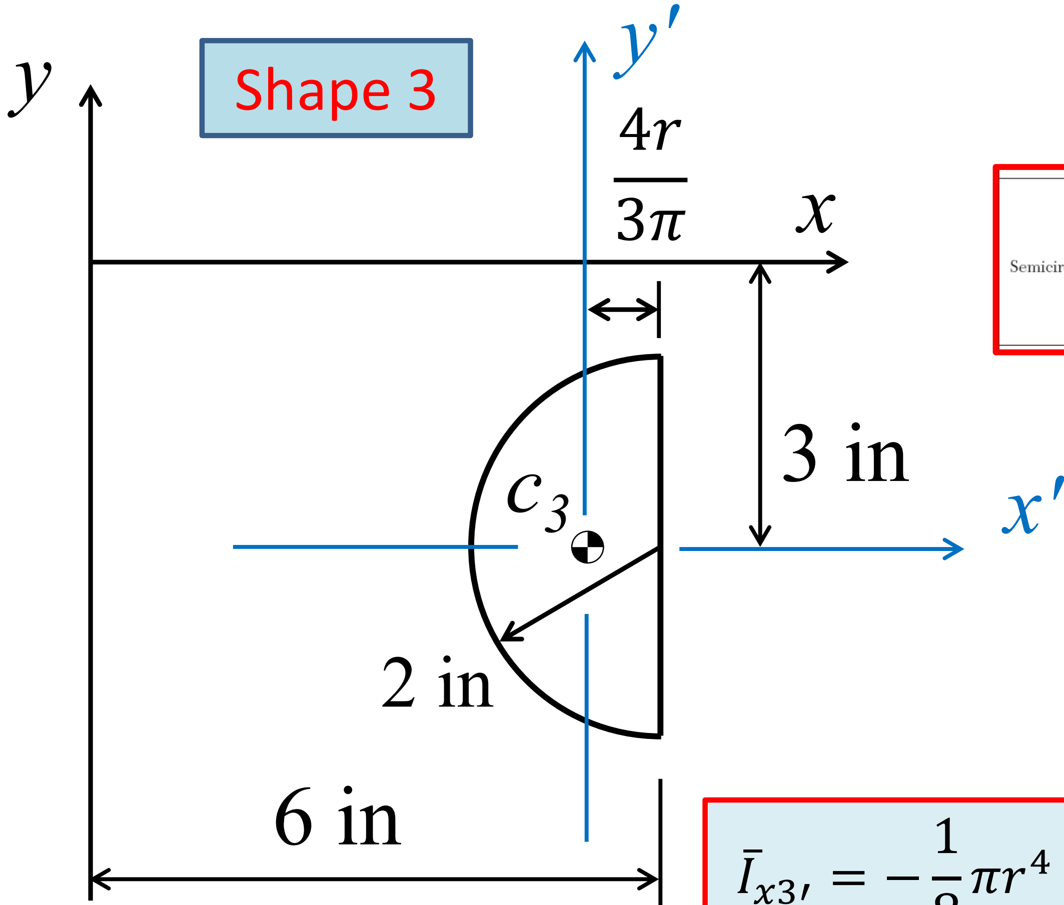
$$I_x = \bar{I}_{x'} + d_y^2 A$$

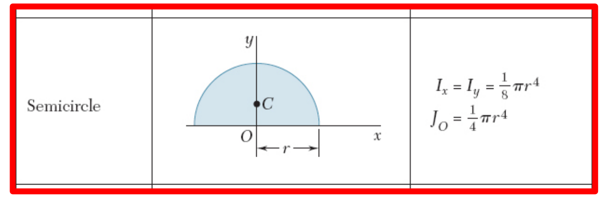
$$\bar{I}_{x'} = I_x - d_y^2 A$$

$$\bar{I}_{x'} = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right) r^4$$

$$\bar{I}_{x'} = \frac{1}{8}\pi r^4 - \left(\frac{4r}{3\pi}\right)^2 \left(\frac{\pi r^2}{2}\right)$$

Find Area and Location of Centroid of Each Shape Relative to Reference Coordinate Axes





$$\bar{I}_{x3'} = -\frac{1}{8}\pi r^4 = -\frac{1}{8}\pi(2^4) = -6.28318\ \text{in}^4$$

$$\bar{I}_{y3'} = -\left(\frac{\pi}{8} - \frac{8}{9\pi}\right) r^4 = -\left(\frac{\pi}{8} - \frac{8}{9\pi}\right)(2^4) = -1.75610\ \text{in}^4$$

Find the Moment of Inertia about the *x* axis

$\bar{I}_{x1'} = 5.25 \text{ in}^4$

$\bar{I}_{x2'} = 21.333 \text{ in}^4$

$\bar{I}_{x3'} = -6.28318 \text{ in}^4$

$d_{y1} = -2.0 \text{ in}$

$d_{y2} = -5.0 \text{ in}$

$d_{y3} = -3 \text{ in}$

$A_1 = 10.5 \text{ in}^2$

$A_2 = 16 \text{ in}^2$

$A_3 = -6.2832 \text{ in}^2$

$$I_x = \sum \bar{I}_{xi'} + \sum d_{yi}^2 A_i$$

Shape	$\bar{I}_{xi'}$	d_{yi}	A_i	$d_{yi}^2 A_i$
1	5.25	–2.0	10.5	42.0
2	21.333	–5.0	16.0	400.0
3	–6.28318	–3.0	-6.28318	–56.5486
$\sum$	20.300			385.4514

$$I_x = 20.30 + 385.4514 = 405.75 \text{ in}^4$$

Find the Moment of Inertia about the y axis

$\bar{I}_{y1'} = 28.5833 \text{ in}^4$

$\bar{I}_{y2'} = 21.333 \text{ in}^4$

$\bar{I}_{y3'} = -1.75610 \text{ in}^4$

$d_{x1} = 4.67 \text{ in}$

$d_{x2} = 5.0 \text{ in}$

$d_{x3} = 5.1512 \text{ in}$

$A_1 = 10.5 \text{ in}^2$

$A_2 = 16 \text{ in}^2$

$A_3 = -6.2832 \text{ in}^2$

$$I_y = \sum \bar{I}_{yi'} + \sum d_{xi}^2 A_i$$

Shape	$\bar{I}_{yi'}$	d_{xi}	A_i	$d_{xi}^2 A_i$
1	28.5833	4.67	10.5	919.0274
2	21.3333	5.0	16.0	400.0
3	–1.75610	5.1512	-6.28318	–166.7233
$\sum$	48.1606			1152.3041

$$I_y = 48.1606 + 1152.3041 = 1200.46 \text{ in}^4$$