Pure Bending

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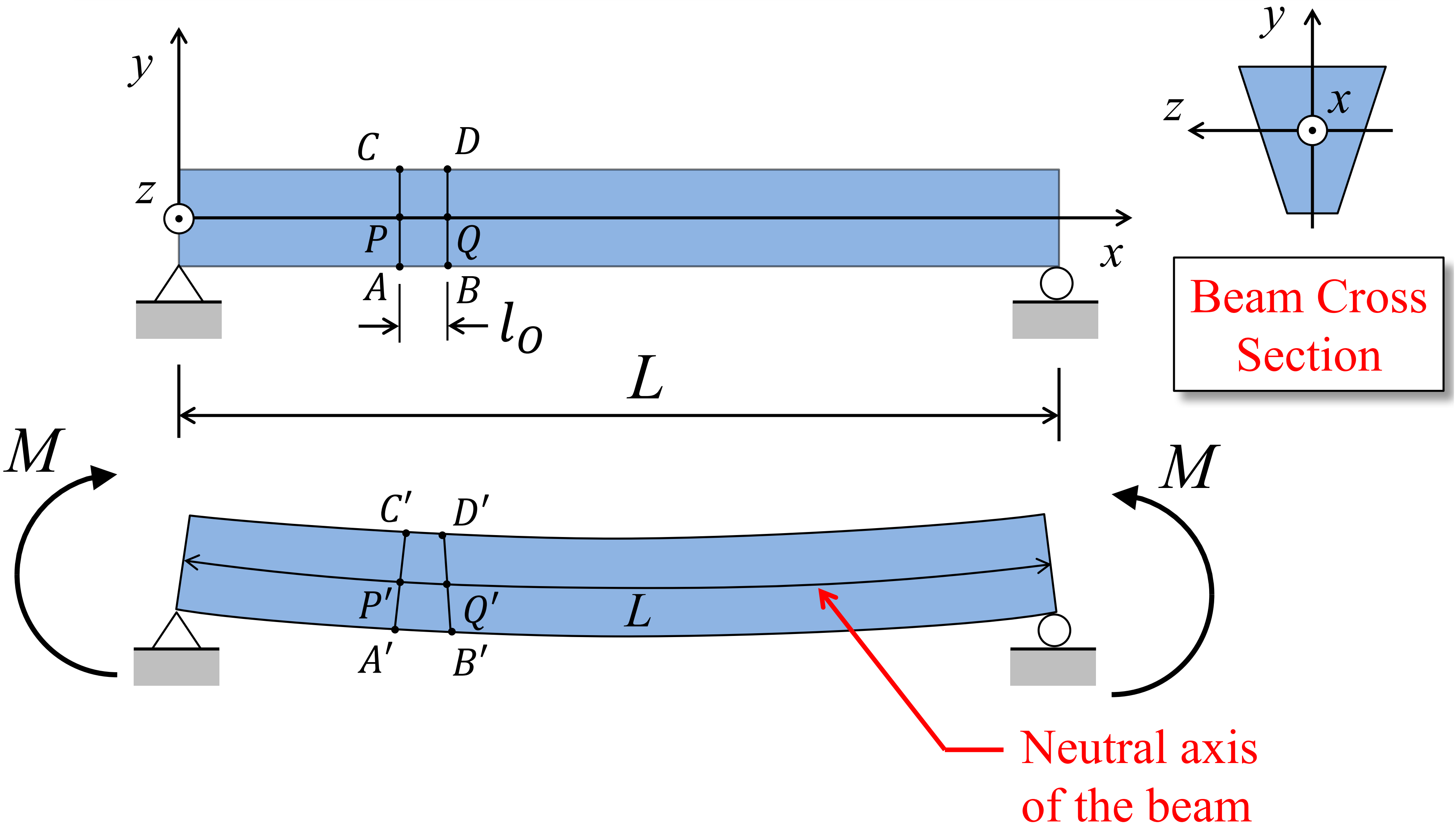
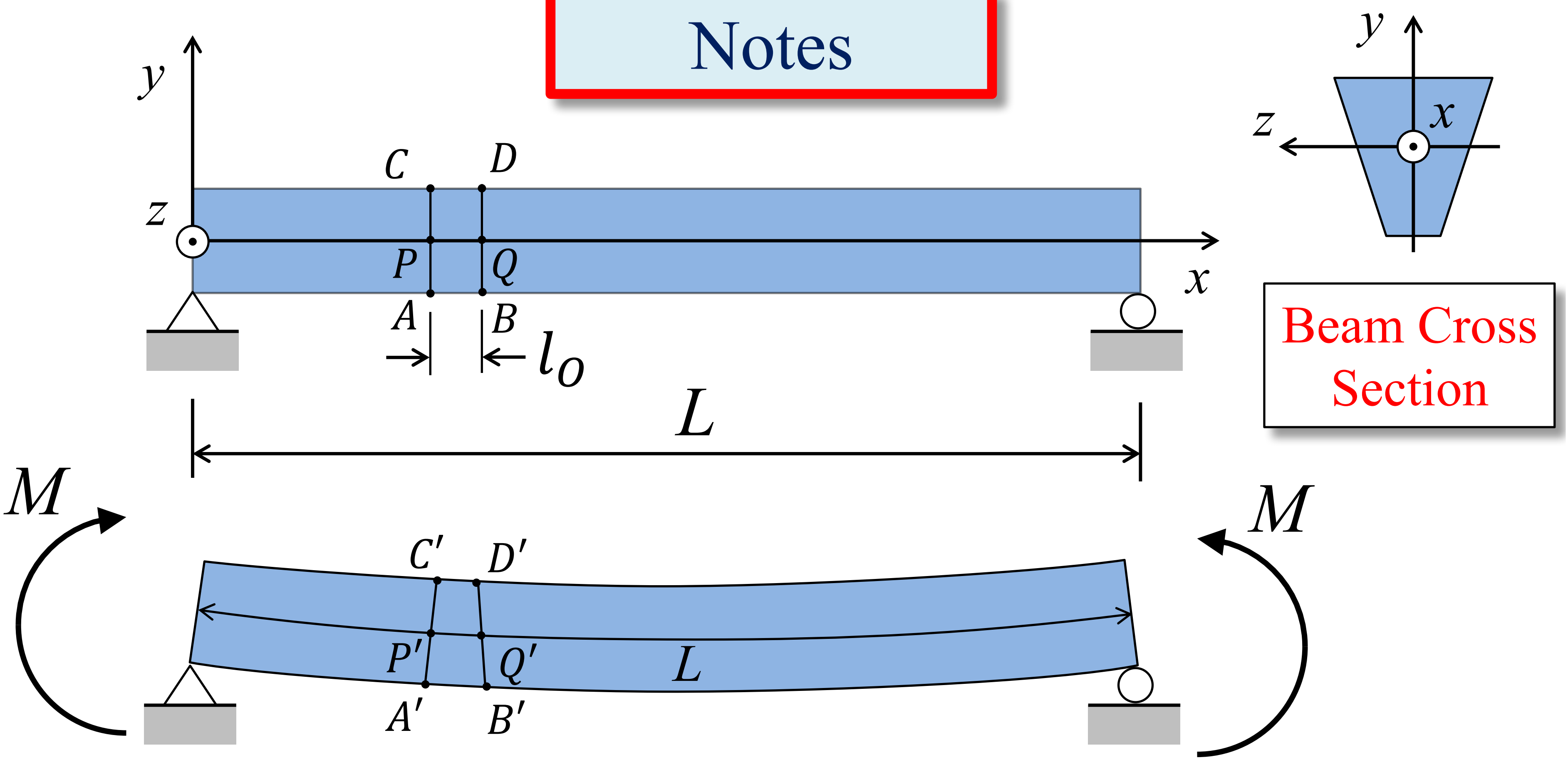
Consider an Elastic Beam Subjected to Pure Bending

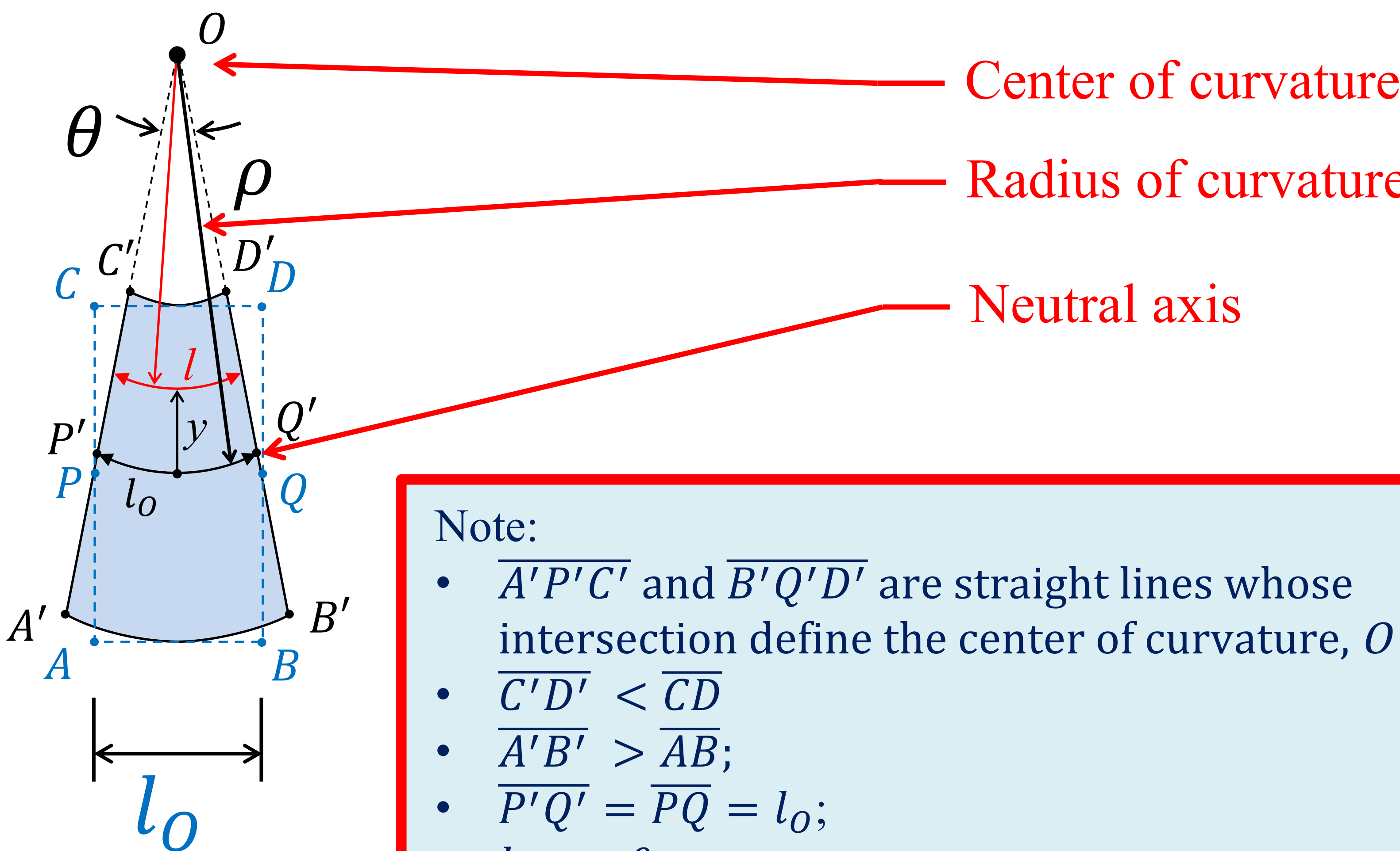


Notes



1. Beam is prismatic and symmetric about the y axis:
2. Material is linear elastic;
3. The x axis is attached to the neutral axis of the beam;
4. Pure bending (no internal shear) –the beam deforms in a circular arc.

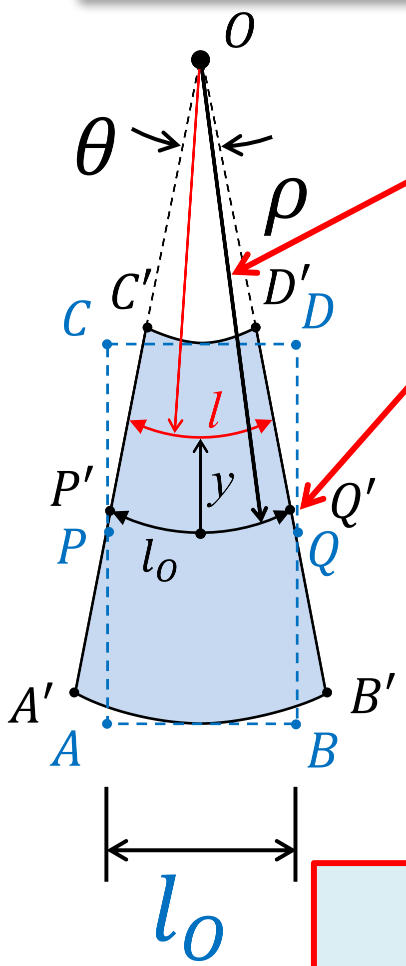
Study the Geometry of the Deformed Shape of a Small Slice of the Beam



Note:

- $\overline{A'P'C'}$ and $\overline{B'Q'D'}$ are straight lines whose intersection define the center of curvature, O;
- $\overline{C'D'} < \overline{CD}$
- $\overline{A'B'} > \overline{AB}$;
- $\overline{P'Q'} = \overline{PQ} = l_o$;
- $l_o = \rho\theta$;
- $l = (\rho - y)\theta$

Bending Strain of the Horizontal Fibers of the Beam



Radius of curvature

Neutral axis

Original length of all fibers, l_O

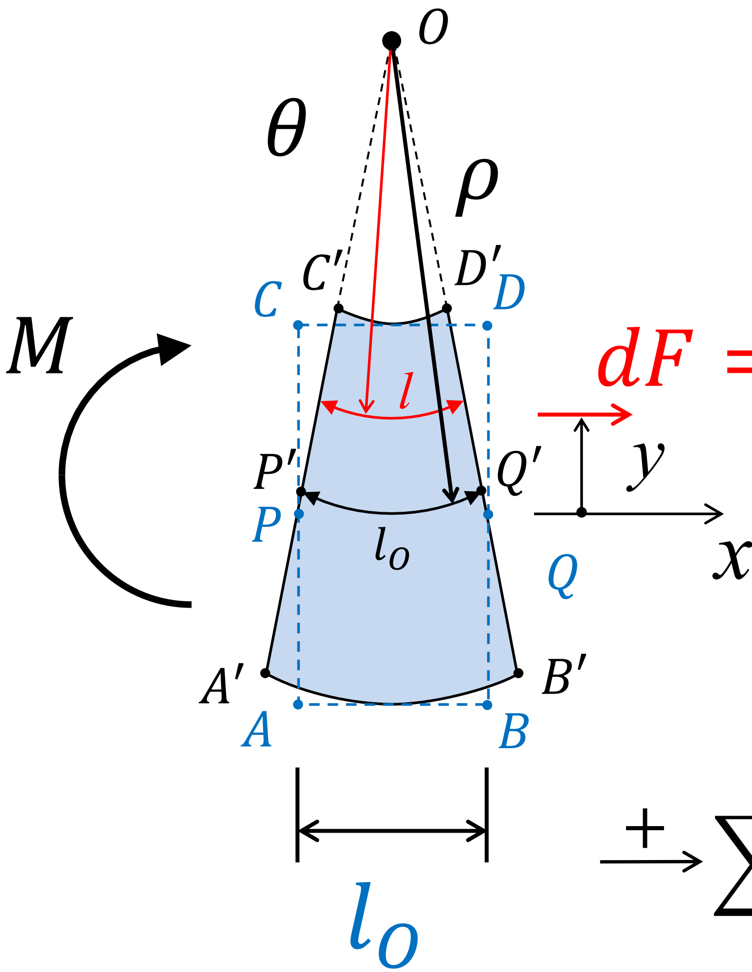
$$l_O = \rho\theta$$

Length of fiber, l, after deformation

$$l = (\rho - y)\theta$$

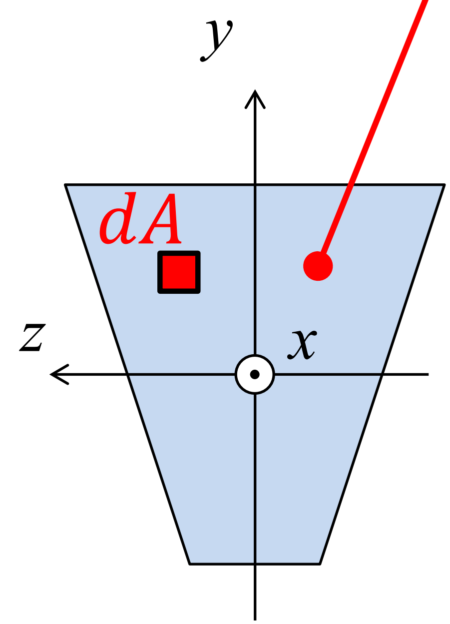
$$\epsilon = \frac{\Delta l}{l_O} = \frac{l - l_O}{l_O} = \frac{(\rho - y)\theta - \rho\theta}{\rho\theta} = -\frac{y}{\rho}$$

Equilibrium of the Small Segment of Beam



$$dF = \sigma dA$$

Beam cross sectional area = A

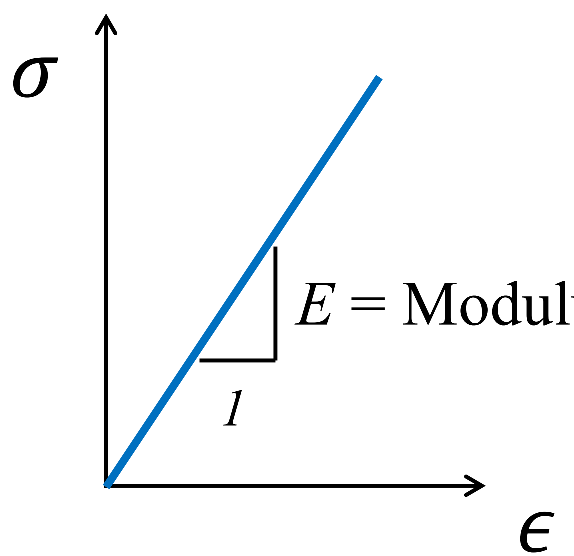


$$\xrightarrow{+} \sum F_x = 0 \qquad \iint_A dF = 0 \qquad \iint_A \sigma dA = 0$$

$$\circlearrowleft + \sum M_z = 0 \qquad -\iint_A y\,dF - M = 0 \qquad -\iint_A y\sigma dA - M = 0$$

Constitutive Law for Beam Material

Beam is made of linear elastic material



$$\sigma = E\epsilon$$

Review Relationships from Geometry of Deformation, Equilibrium, and Constitutive Law

1. $$\epsilon = -\frac{y}{\rho}$$

2. $$\iint_A \sigma dA = 0$$ $$-\iint_A y\sigma dA - M = 0$$

3. $$\sigma = E\epsilon$$

Substituting equation 1 into equation 3

$$\sigma = -E\left(\frac{y}{\rho}\right)$$

Force Equilibrium in x Direction

$$\iint_A \sigma dA = 0$$

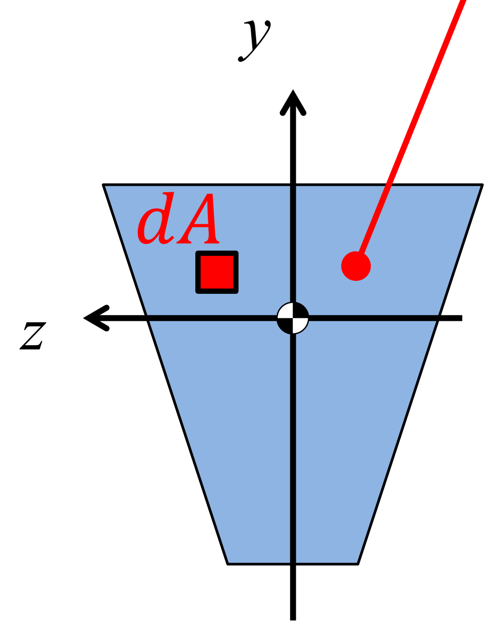
$$\sigma = -E\left(\frac{y}{\rho}\right)$$

$$\iint_A -E\left(\frac{y}{\rho}\right) dA = 0$$

$$-\frac{E}{\rho}\iint_A y dA = 0$$

$$\iint_A y dA = 0$$

Beam cross sectional area = A



Can only be satisfied if the neutral axis is at the **centroid** of the beam cross section

Moment Equilibrium

$$-\iint_A y\sigma dA - M = 0$$

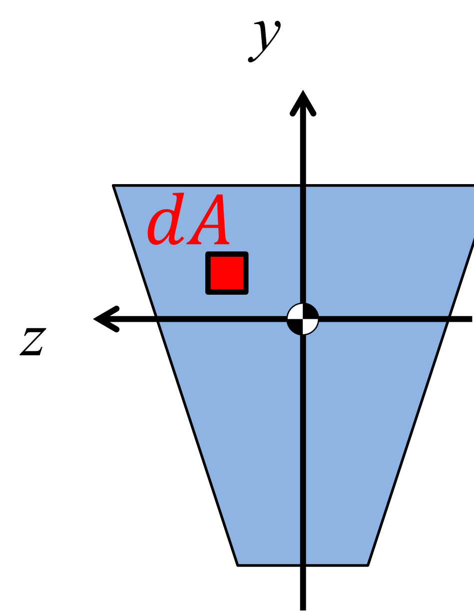
$$\sigma = -E\left(\frac{y}{\rho}\right)$$

$$-\iint_A -yE\left(\frac{y}{\rho}\right) dA - M = 0$$

$$\frac{E}{\rho}\iint_A y^2 dA - M = 0$$

$$M = \frac{E}{\rho}\iint_A y^2 dA$$

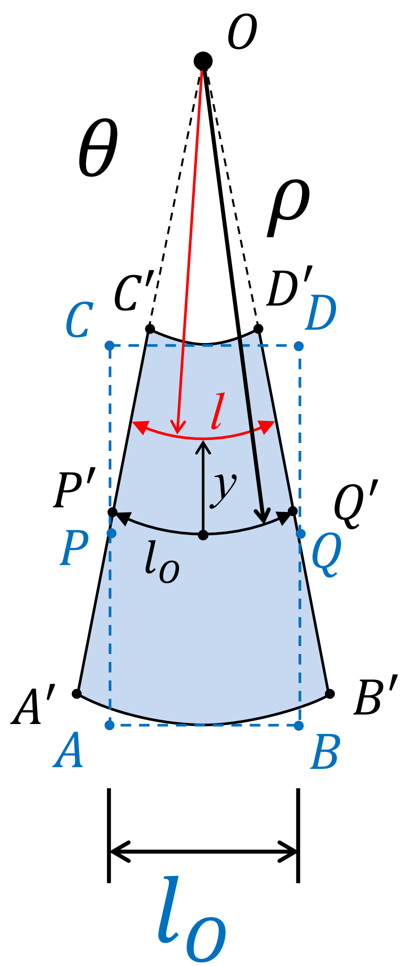
$$M = \frac{EI}{\rho}$$



Moment of Inertia about the centroid of the beam cross section

$$I = \iint_A y^2 dA$$

Moment-Curvature Relationship



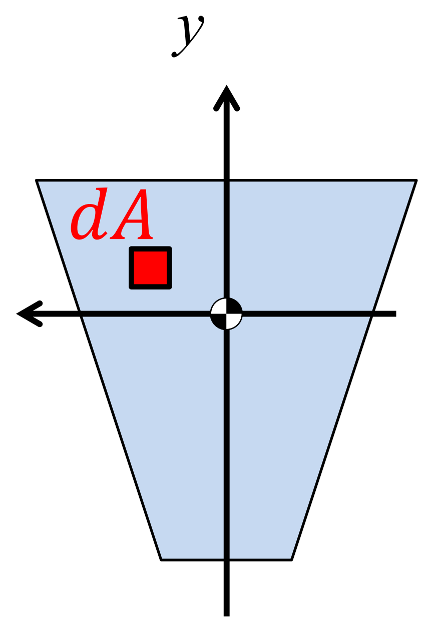
$$M = \frac{EI}{\rho}$$

Define:

κ = Curvature of the beam

$$\kappa = \frac{1}{\rho}$$

$$\kappa = \frac{M}{EI}$$

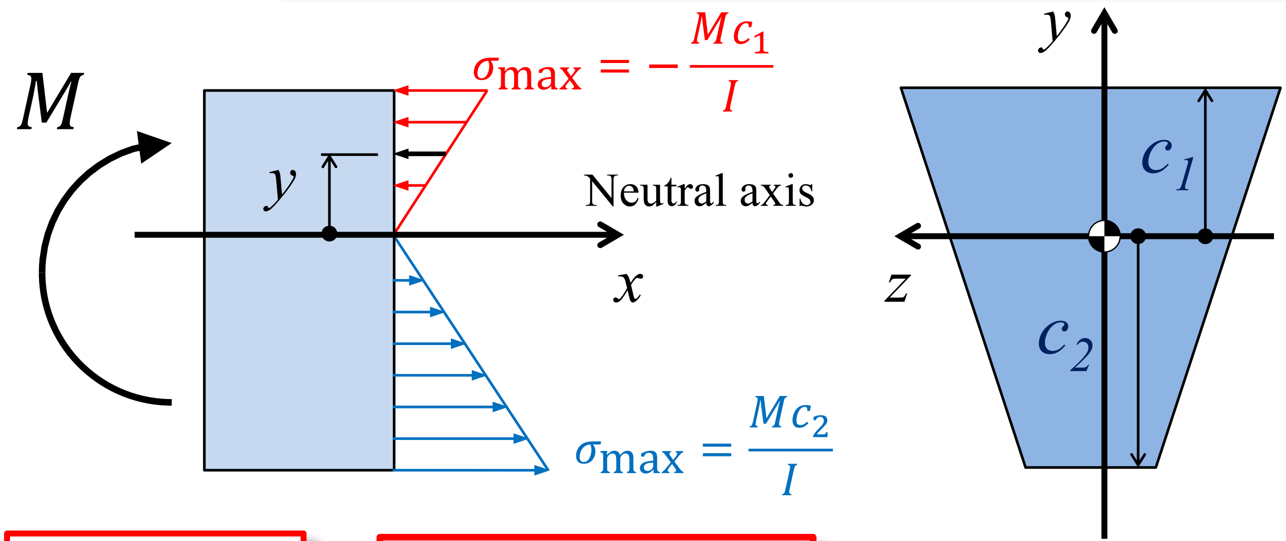


Moment of Inertia about the centroid of the beam cross section

$$I = \iint_A y^2 dA$$

The moment-curvature relationship is the basis of bending deformation theory

Bending Stress Distribution



$$y_{max} = c$$

$$M = \frac{EI}{\rho}$$

$$\sigma = -E\left(\frac{y}{\rho}\right)$$

$$\frac{1}{\rho} = \frac{M}{EI}$$

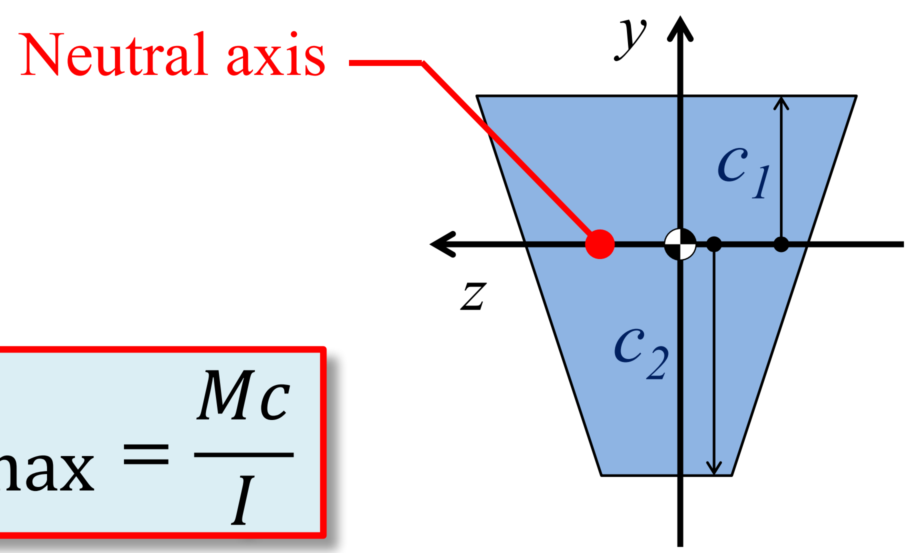
$$\sigma = -Ey\left(\frac{1}{\rho}\right) = -Ey\left(\frac{M}{EI}\right)$$

$$\sigma = -\frac{My}{I}$$

Summary for Pure Bending of an Elastic Beam

Moment-Curvature relationship

$$\kappa = \frac{M}{EI}$$



Bending stress distribution

$$\sigma = -\frac{My}{I}$$

$$\sigma_{\max} = \frac{Mc}{I}$$

1. Neutral axis ($\sigma = 0$) is located at the centroid of the beam cross section;
2. Moment-Curvature relationship is basis of bending deformation theory;
3. Bending stress varies linearly over beam cross section and is maximum at the extreme fibers of the beam;