Method of Virtual Work
Beam Deflection Example

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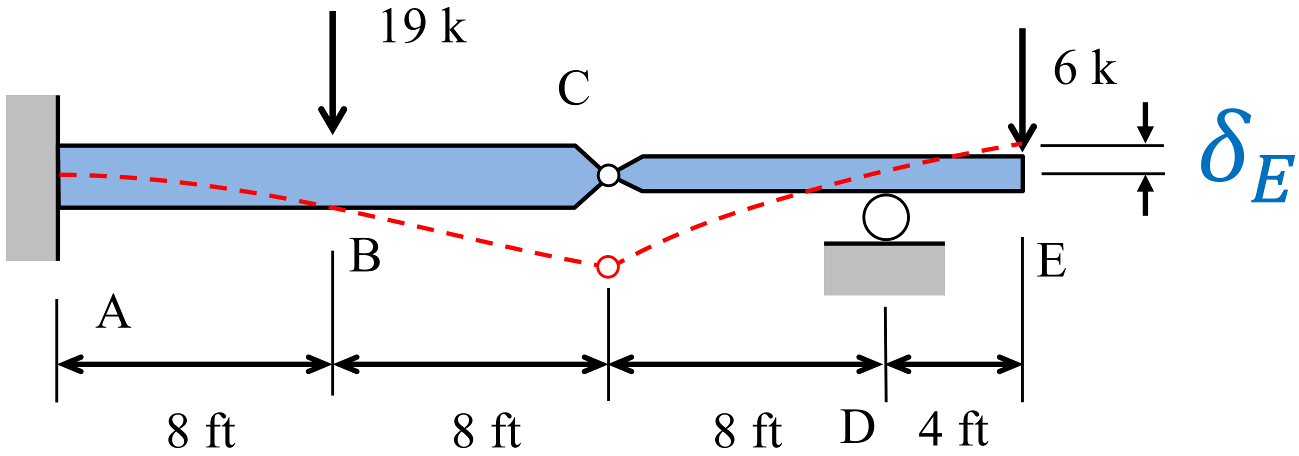
Beam Deflection Example

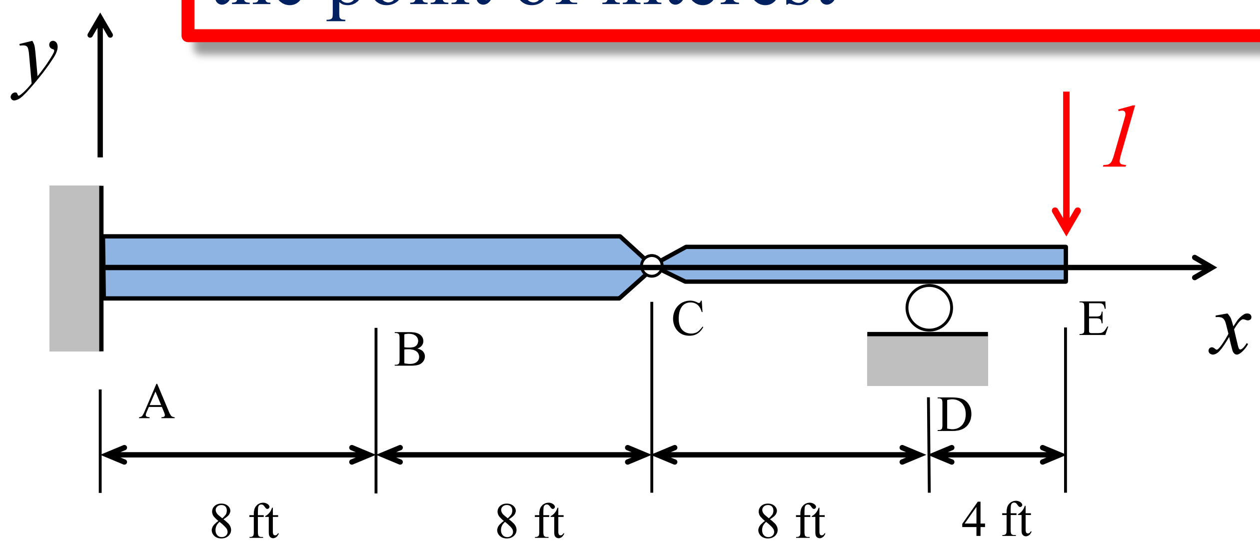


The overhanging beam shown has a fixed support at A, a roller support at C and an internal hinge at B. EI_{ABC} = 2,000,000 k-in^2 and EI_{CDE} = 800,000 k-in^2

For the loads shown, find the vertical deflection at point E.

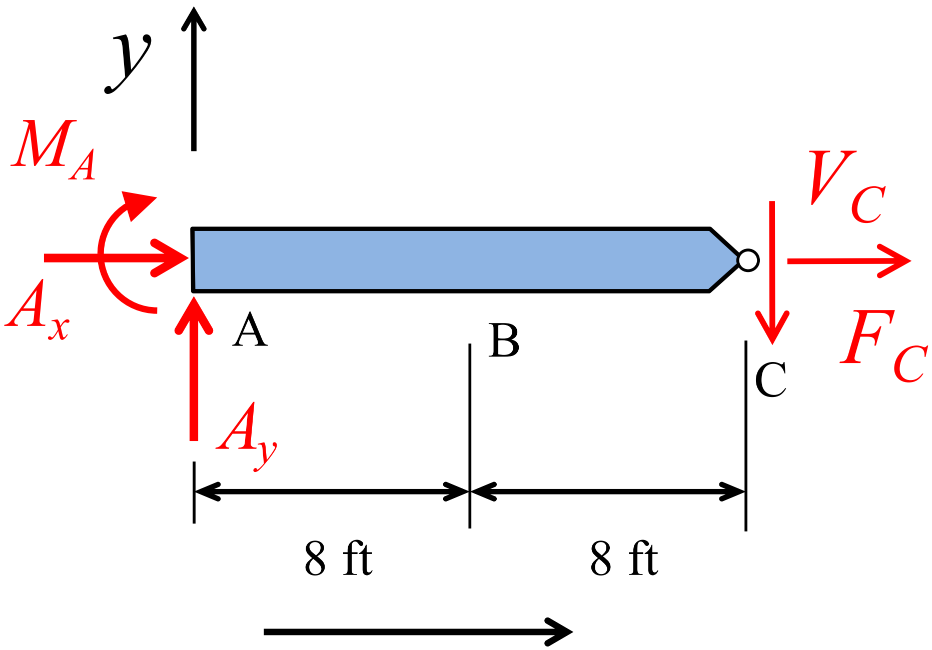
Find the Deflection at Point E

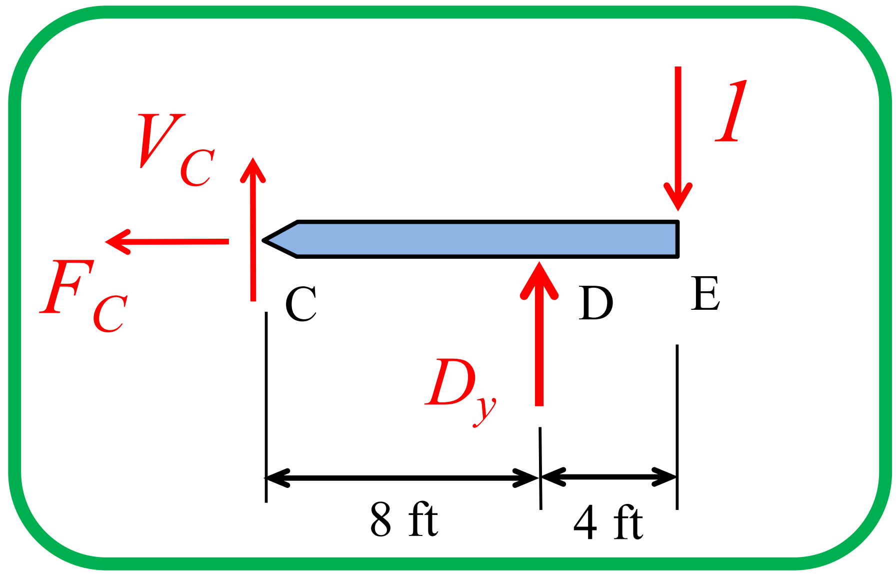
Step 1 – Remove all loads and apply a virtual force (or moment) to measure the deformation at the point of interest



From an equilibrium analysis, find the internal bending moment function for the virtual system: $M_Q(x)$

Find the Moment Functions for the Virtual System





$\circlearrowleft + \sum M_A = 0 \rightarrow$ $M_A = 8\text{ ft}$

$\circlearrowleft + \sum M_C = 0 \rightarrow$ $D_y = 1.5$

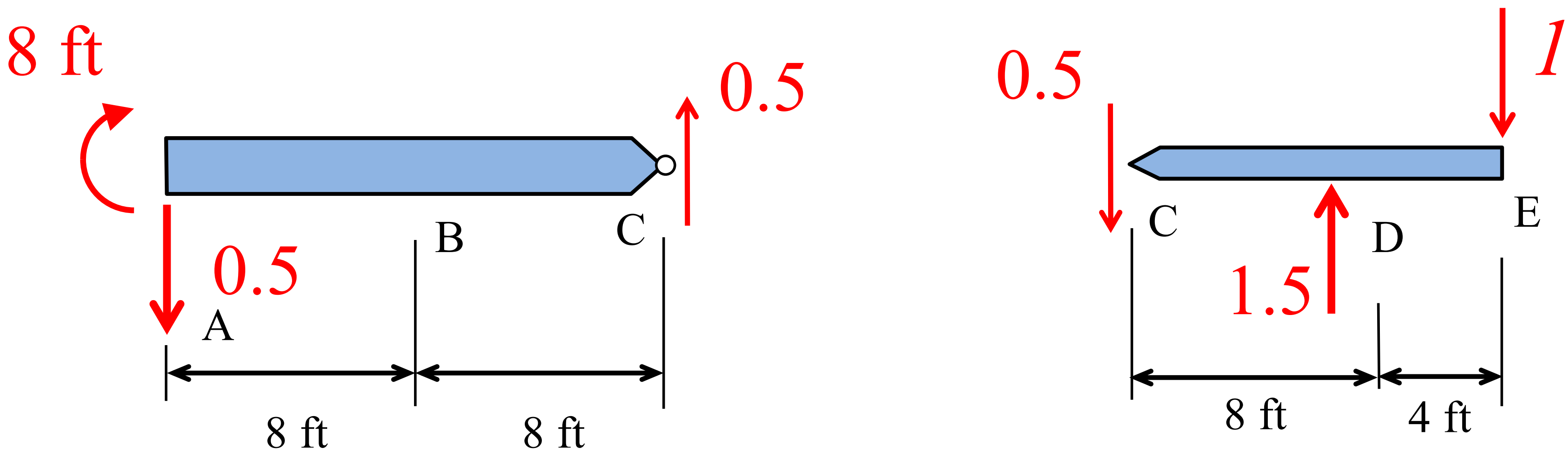
$\xrightarrow{+} \sum F_x = 0 \rightarrow$ $A_x = 0$ ⇦

$\xrightarrow{+} \sum F_x = 0 \rightarrow$ $F_C = 0$

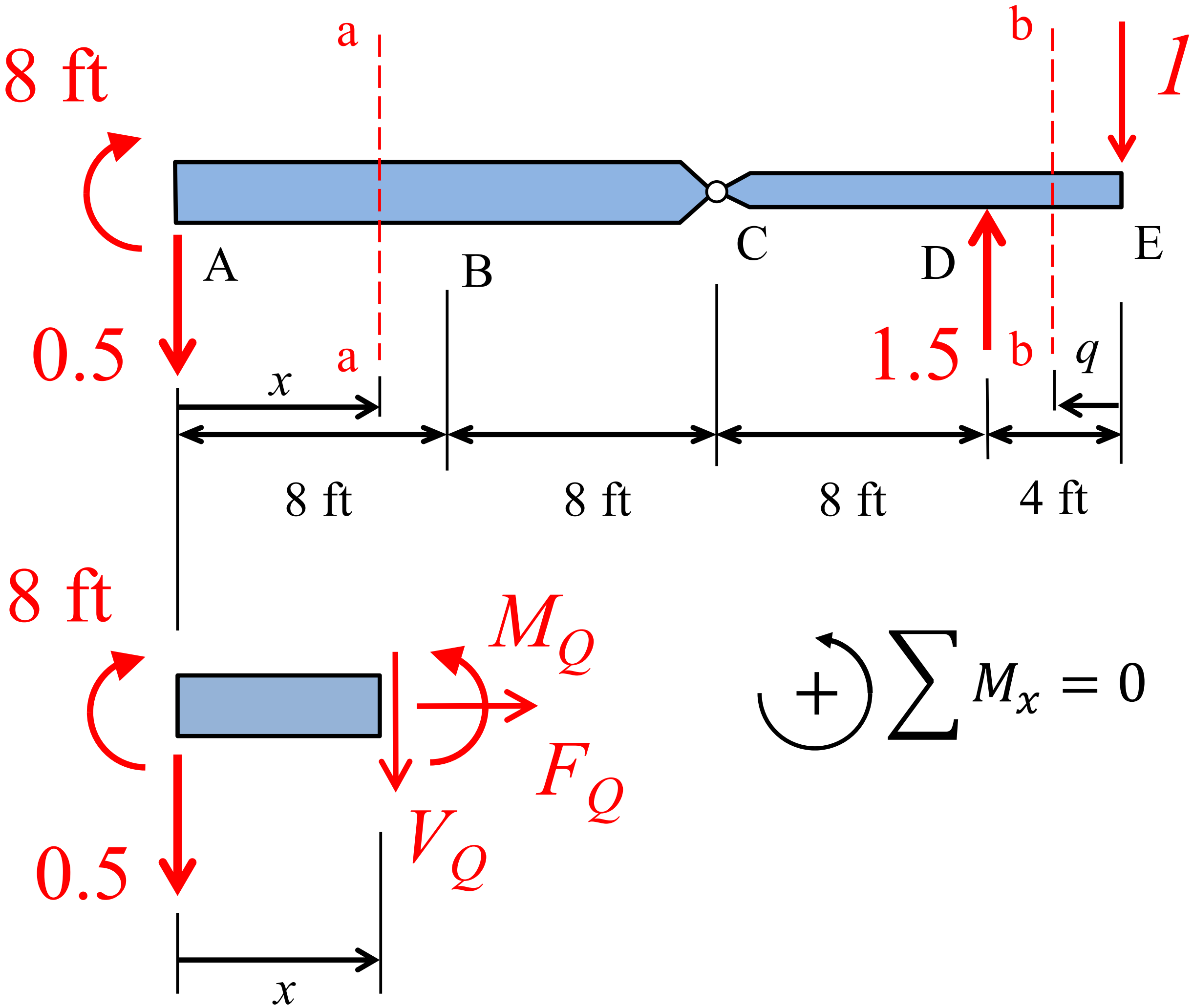
$+\uparrow \sum F_y = 0 \rightarrow$ $A_y = -0.5$

$+\uparrow \sum F_y = 0 \rightarrow$ $V_C = -0.5$

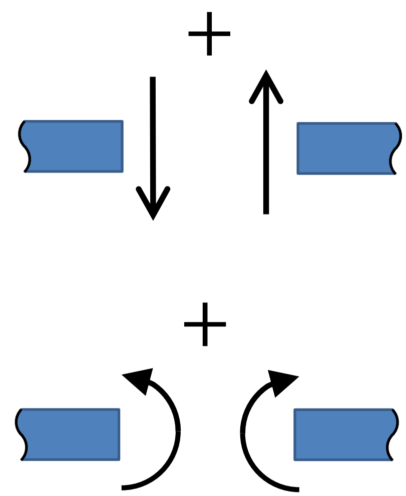
Support Reactions for the Virtual System



Moment Functions for the Virtual System

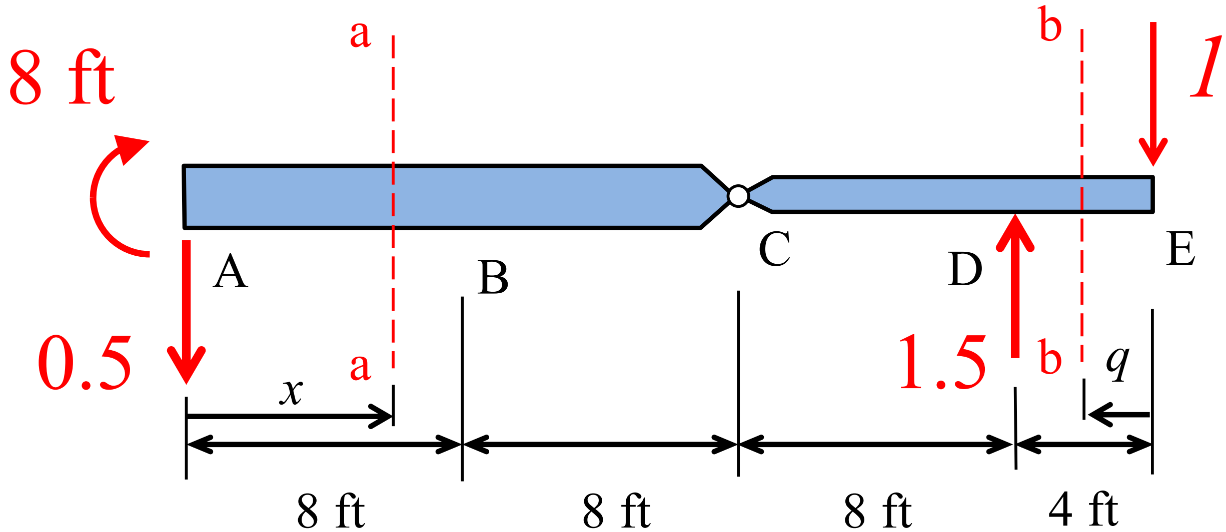


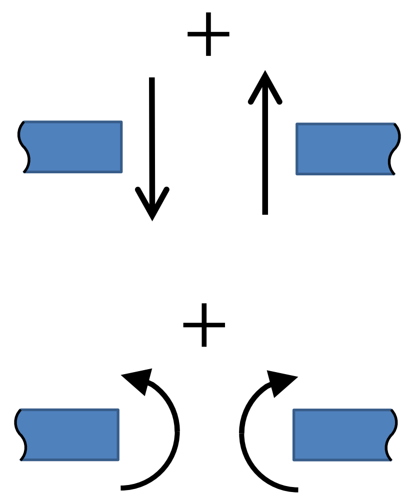
$$(+\circlearrowleft)\ \sum M_x = 0$$



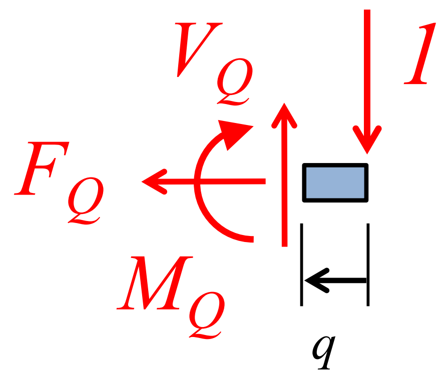
$$M_Q = 8 - 0.5x \qquad 0 < x < 24$$

Moment Functions for the Virtual System



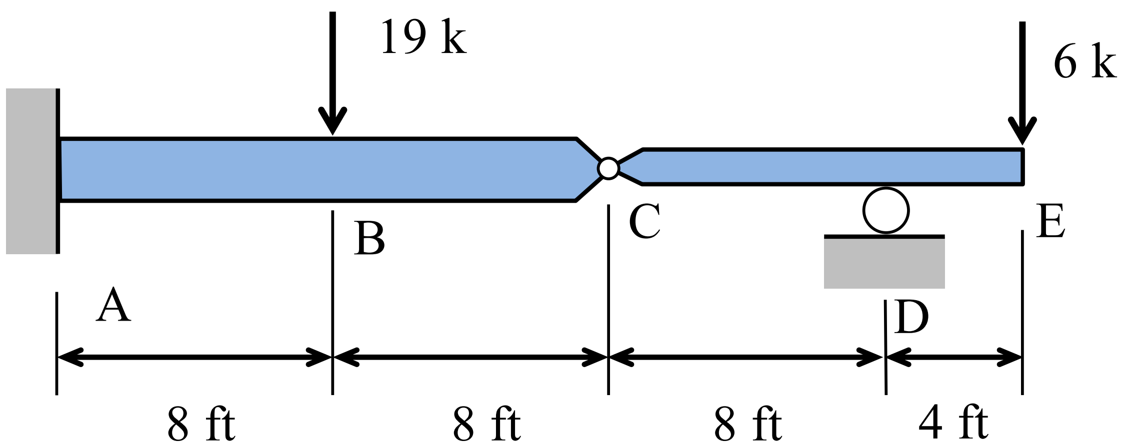


$$\circlearrowleft + \sum M_q = 0$$



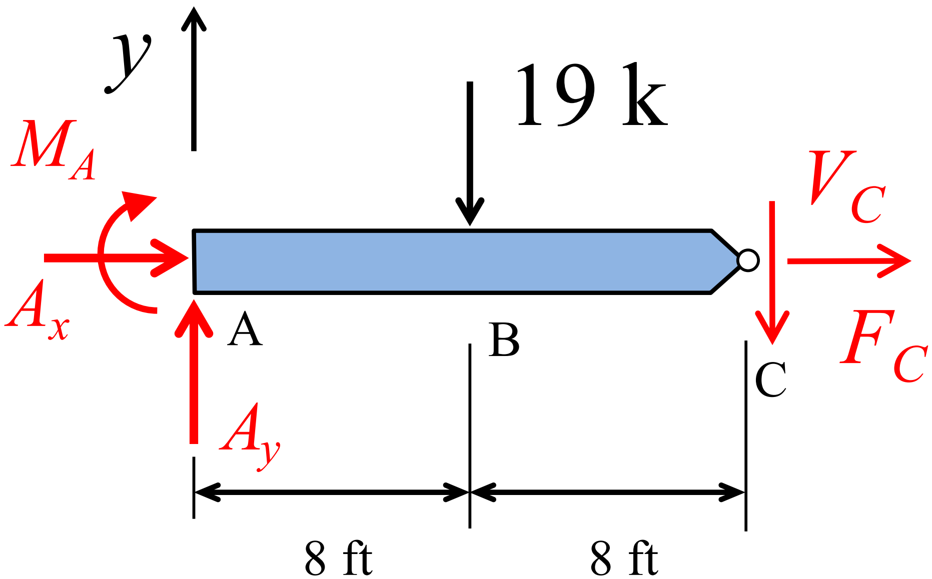
$$M_Q = -q \qquad 0 < q < 4$$

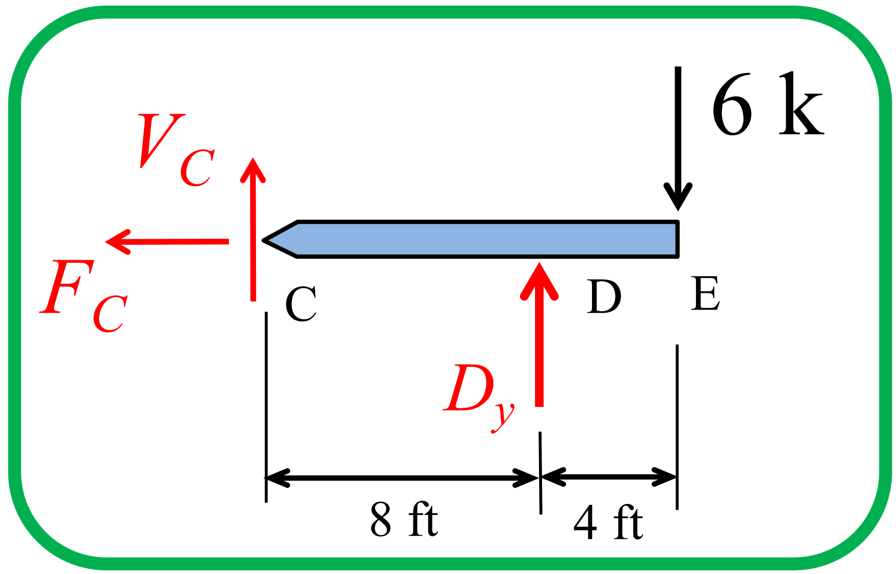
Step 2 – Replace all of the loads on the structure and perform the real analysis



From an equilibrium analysis, find the internal bending moment function for the real system: $M_P(x)$

Find the Moment Diagram for the Real System





$\circlearrowleft + \sum M_A = 0 \rightarrow$ $M_A = -104$ k-ft

$\xrightarrow{+} \sum F_x = 0 \rightarrow$ $A_x = 0$

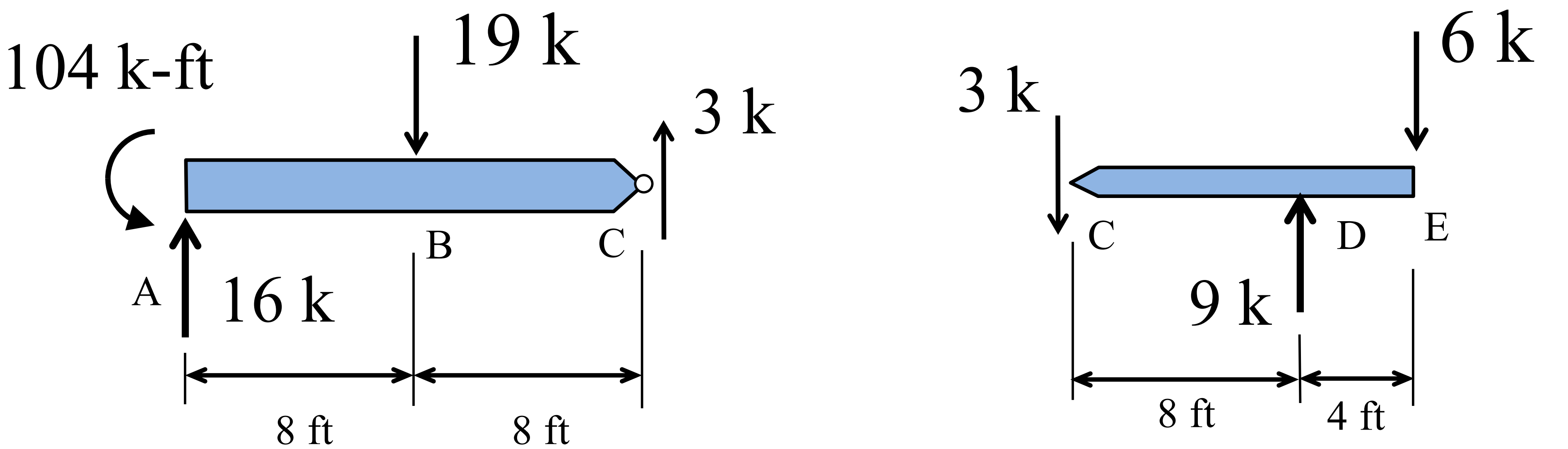
$+\uparrow \sum F_y = 0 \rightarrow$ $A_y = 16$ k

$\circlearrowleft + \sum M_C = 0 \rightarrow$ $D_y = 9$ k

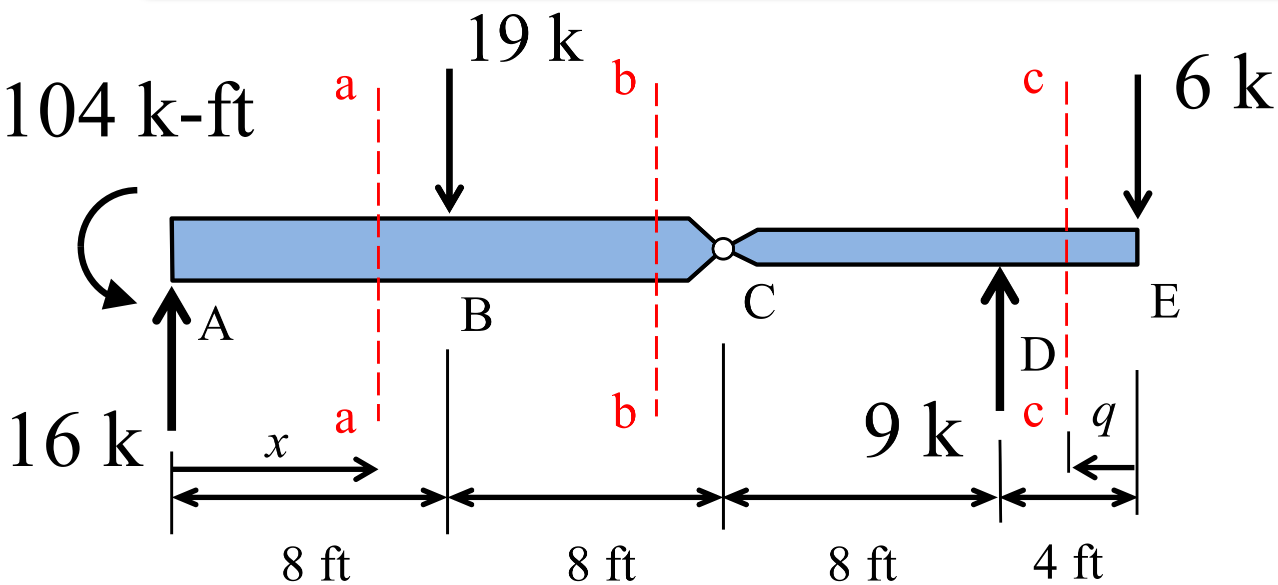
$\xrightarrow{+} \sum F_x = 0 \rightarrow$ $F_C = 0$

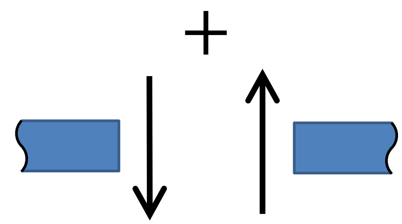
$+\uparrow \sum F_y = 0 \rightarrow$ $V_C = -3$ k

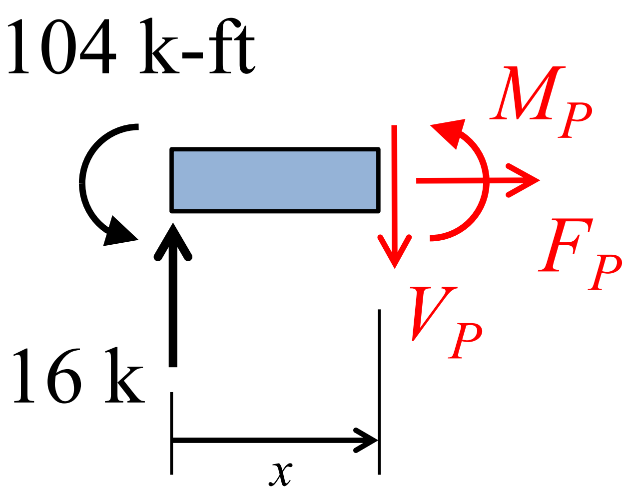
Support Reactions for the Real System



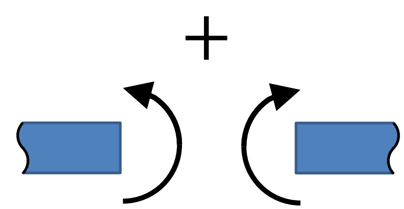
Moment Functions for the Real System





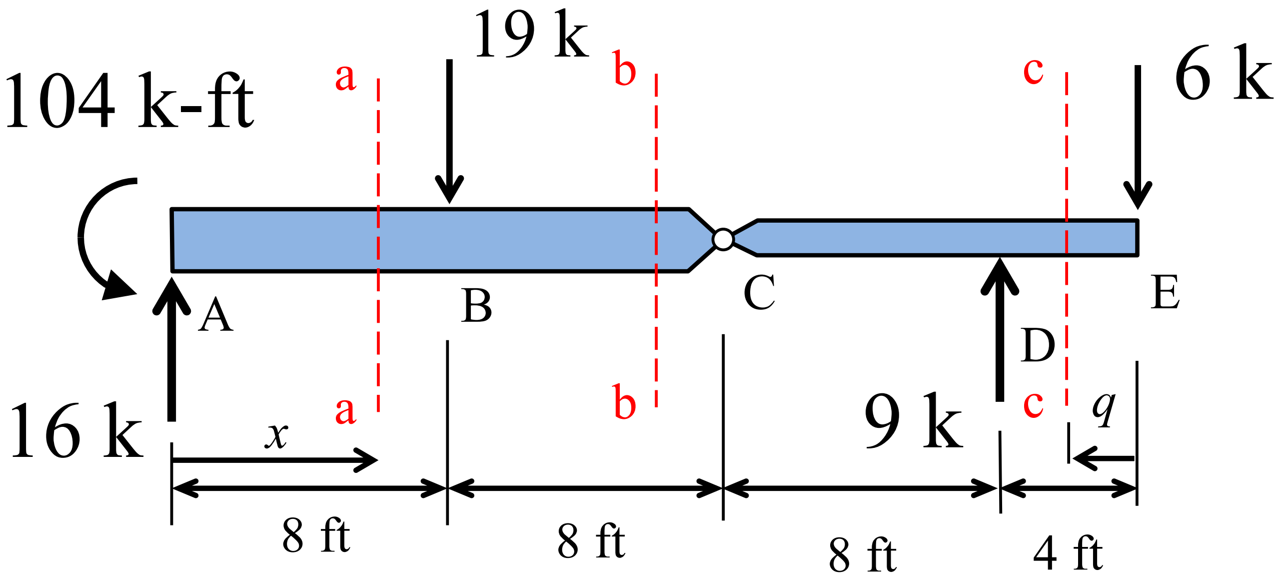


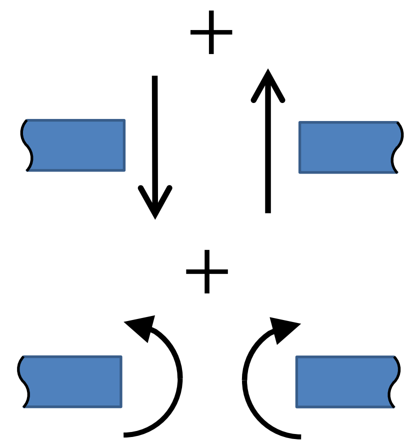
$$\circlearrowleft + \sum M_x = 0$$

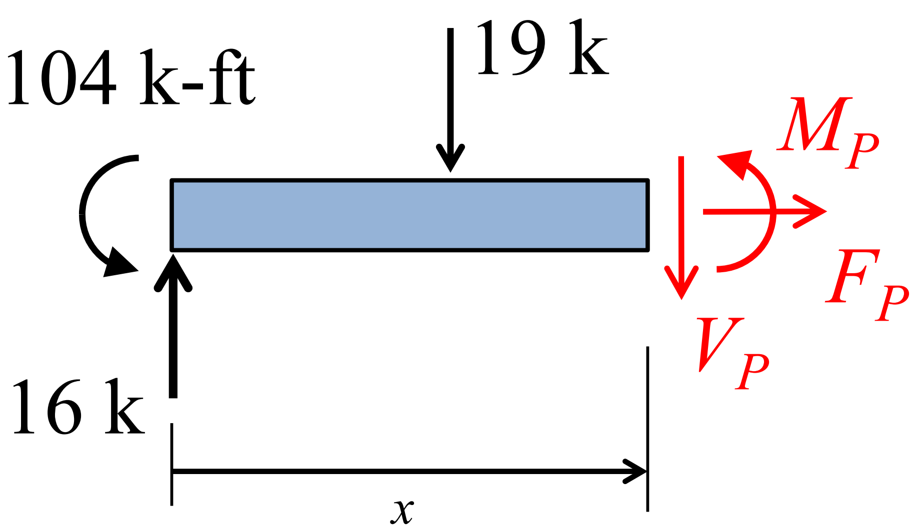


$$M_P = -104 + 16x \quad 0 < x < 8$$

Moment Functions for the Real System



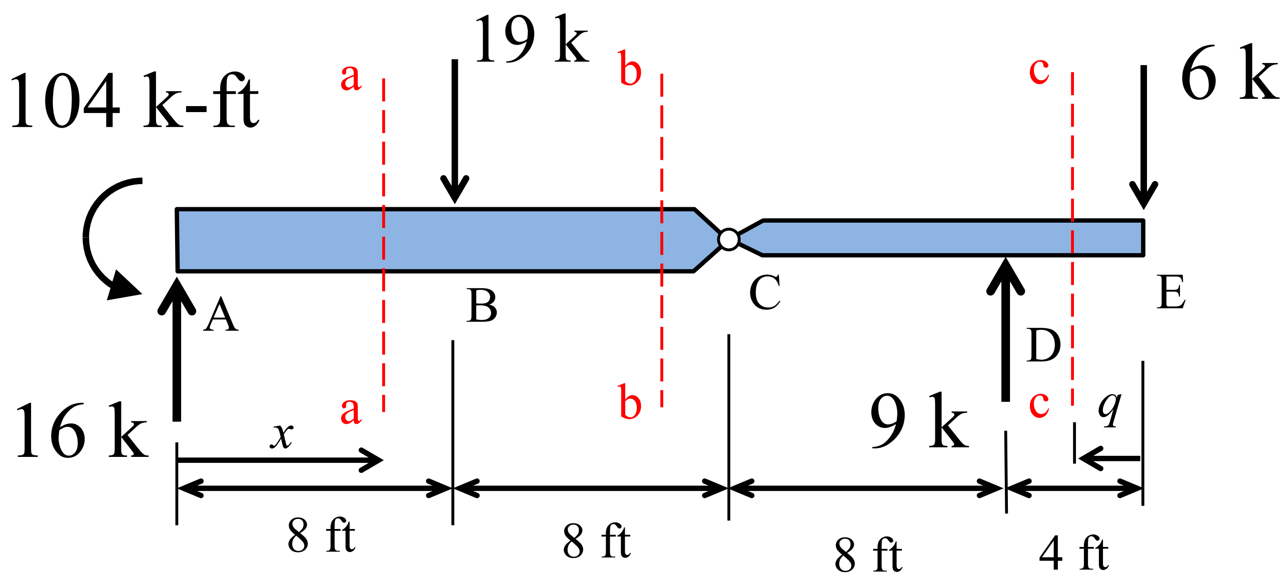


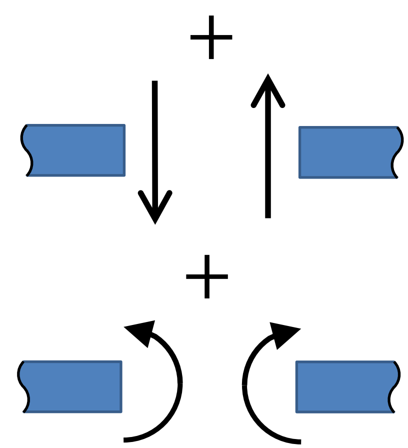


$$\circlearrowleft + \sum M_x = 0$$

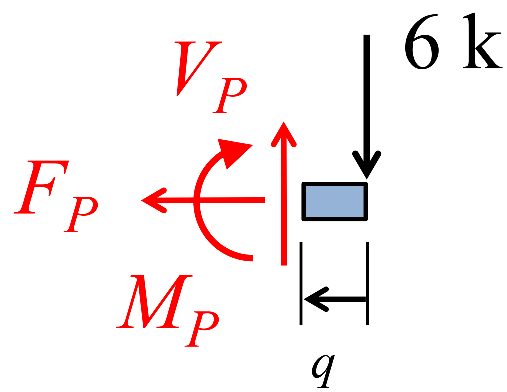
$$M_P = 48 - 3x \qquad 8 < x < 24$$

Moment Functions for the Real System





$$+\circlearrowleft \sum M_q = 0$$



$$M_P = -6q \qquad 0 < q < 4$$

Evaluate Product Integrals for ABC

$$\int_0^{L_{ABC}} M_Q\, M_P dx = \int_0^{8} (8 - 0.5x)\,(-104 + 16x)dx + \int_8^{16} (8 - 0.5x)\,(48 - 3x)dx$$

$$\int_0^{L_{ABC}} M_Q\, M_P dx = \int_0^{8} (-832 + 180x - 8x^2)\, dx + \int_8^{16} (384 - 48x + 1.5x^2)\, dx$$

$$\int_0^{L_{ABC}} M_Q\, M_P dx = \left[-832x + 180\frac{x^2}{2} - 8\frac{x^3}{3}\right]_0^8 + \left[384x - 48\frac{x^2}{2} + 1.5\frac{x^3}{3}\right]_8^{16}$$

$$\int_0^{L_{ABC}} M_Q\, M_P dx = \left[-832(8) + 180\frac{(8)^2}{2} - 8\frac{(8)^3}{3}\right]$$
$$+ \left[384(16) - 48\frac{(16)^2}{2} + 1.5\frac{(16)^3}{3}\right] - \left[384(8) - 48\frac{(8)^2}{2} + 1.5\frac{(8)^3}{3}\right]$$

$$\int_0^{L_{ABC}} M_Q\, M_P dx = [-2261.333] + [2048] - [1792] = \mathbf{-2005.333\ k\text{–}ft^3}$$

Evaluate Product Integrals for CDE

$$\int_0^{L_{CDE}} M_Q\, M_P\, dx = \int_{16}^{24} (8 - 0.5x)(48 - 3x)dx + \int_0^4 (-q)(-6q)dq$$

$$\int_0^{L_{CDE}} M_Q\, M_P\, dx = \int_{16}^{24} (384 - 48x + 1.5x^2)\, dx + \int_0^4 (6q^2)dq$$

$$\int_0^{L_{CDE}} M_Q\, M_P\, dx = \left[384x - 48\frac{x^2}{2} + 1.5\frac{x^3}{3}\right]_{16}^{24} + \left[6\frac{q^3}{3}\right]_0^4$$

$$\int_0^{L_{CDE}} M_Q\, M_P\, dx = \left[384(24) - 48\frac{(24)^2}{2} + 1.5\frac{(24)^3}{3}\right] - \left[384(16) - 48\frac{16^2}{2} + 1.5\frac{(16)^3}{3}\right] + \left[6\frac{(4)^3}{3}\right]$$

$$\int_0^{L_{CDE}} M_Q\, M_P\, dx = [2304] - [2048] + [128] = \mathbf{384\ k\text{–}ft^3}$$

Calculate Total Internal Work

$$1 \cdot \delta_E = \frac{1}{EI_{ABC}} \int_0^{L_{ABC}} M_Q\, M_P\, dx + \frac{1}{EI_{CDE}} \int_0^{L_{CDE}} M_Q\, M_P\, dx$$

$$\int_0^{L_{ABC}} M_Q\, M_P\, dx = (-2005.333 \text{ k–ft}^3)\left(\frac{12^3 \text{ in}^3}{\text{ft}^3}\right) = -3{,}465{,}216.0 \text{ k–in}^3$$

$$\int_0^{L_{CDE}} M_Q\, M_P\, dx = (384 \text{ k–ft}^3)\left(\frac{12^3 \text{ in}^3}{\text{ft}^3}\right) = 663{,}552 \text{ k–in}^3$$

$$\delta_E = \frac{-3{,}465{,}216.0 \text{ k–in}^3}{2{,}000{,}000 \text{ k–in}^2} + \frac{663{,}552 \text{ k–in}^3}{800{,}000 \text{ k–in}^2}$$

$$\delta_E = -1.733 \text{ in} + 0.8294 \text{ in} = -\, 0.903 \text{ in}$$

Negative result, so deflection is in the opposite direction of the virtual unit load

$$\delta_E = 0.903 \text{ in upward}$$

Beam Deflection Example Result