Chapter 6. Manipulator Dynamics

11-3-14

Quiz on Nov. 11 on Homework #8

Homework #8. Not collected.

Solve 6.1 (Answer partially given in the textbook). 6.12 (Answer given). 6.16.

Show how (6.32) is derived from (6.15) and (5.45).

Trace the steps taken to derive (6.36) from (6.12).

Verify the formulation of (6.42).

See the Example in Section 6.7 – Two link robot arm with simplifying assumptions. Check the vector cross multiplications at several places in the solution.

Acceleration of Rigid Body – Definition:

Acceleration of linear velocity vector V_Q in frame {B}

$$^{B}\dot{V}s_{Q} = \frac{d}{dt}{}^{B}V_{Q} = \lim_{\Delta t \to 0} \frac{^{B}V_{Q}(t+\Delta t) - {}^{B}V_{Q}(t)}{\Delta t} \tag{6.1}$$

Acceleration of angular velocity vector ω_Q in frame {B}

$$^{A}\dot{\Omega}_{Q} = \frac{d}{dt}{}^{A}\Omega_{Q} = \lim_{\Delta t \to 0} \frac{^{A}\Omega_{Q}(t+\Delta t) - {}^{A}\Omega_{Q}(t)}{\Delta t} \tag{6.2}$$

Linear Acceleration:

From (5.12),

$$^{A}V_{Q} = \frac{d}{dt}({}^{A}_{B}R\,{}^{B}Q) = {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q \tag{6.5}$$

Differentiating (6.5) and a term for linear acceleration of the origin of {B},

$$^{A}\dot{V}_{Q} = \frac{d}{dt}({}^{A}_{B}R\,{}^{B}V_{Q}) + {}^{A}\dot{\Omega}_{B} \times {}^{A}_{B}R\,{}^{B}Q + {}^{A}\Omega_{B} \times \frac{d}{dt}({}^{A}_{B}R\,{}^{B}Q) \tag{6.7}$$

$$= ({}^{A}_{B}R\,{}^{B}\dot{V}_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}V_{Q}) + {}^{A}\dot{\Omega}_{B} \times {}^{A}_{B}R\,{}^{B}Q + {}^{A}\Omega_{B} \times ({}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q) \tag{6.8}$$

With the linear acceleration of $\{B\}_{Orig}$

$$^{A}\dot{V}_{Q} = {}^{A}\dot{V}_{BOrg} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}V_{Q} + 2\,{}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}V_{Q}\ {}^{A}_{B}R\,{}^{B}\dot{V}_{Q} + {}^{A}\dot{\Omega}_{B} \times {}^{A}_{B}R\,{}^{B}Q + {}^{A}\Omega_{B} \times ({}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q)$$

When ^{B}Q is constant,

$$^{A}\dot{V}_{Q} = {}^{A}\dot{V}_{BOrg} + {}^{A}_{B}R\,{}^{B}\dot{V}_{Q} + {}^{A}\dot{\Omega}_{B} \times {}^{A}_{B}R\,{}^{B}Q + {}^{A}\Omega_{B} \times ({}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q) \tag{6.12}$$

Angular Acceleration:

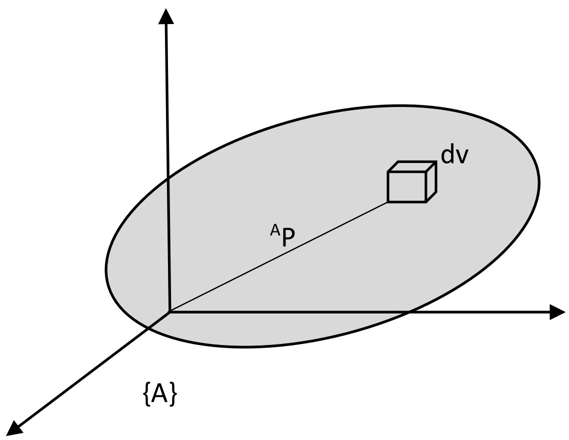
To find the angular acceleration of {C} w.r.t. {A}, differentiate

$$ {}^{A}\Omega_C = {}^{A}\Omega_B + {}^{A}_{B}R\,{}^{B}\Omega_C \tag{6.13} $$

$$ {}^{A}\dot{\Omega}_C = {}^{A}\dot{\Omega}_B + \frac{d}{dt}({}^{A}_{B}R\,{}^{B}\Omega_C) = {}^{A}\dot{\Omega}_B + {}^{A}_{B}R\,{}^{B}\dot{\Omega}_C + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}\Omega_C \tag{6.15} $$

Rigid Body Mass Distribution

Inertia tensor – Describes the distribution of the mass around the center of a rigid body.



${}^{A}P$ is the location vector of the differential volume dv.

Inertia Tensor of {A}:

$$ {}^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} $$

Mass moment of inertia:

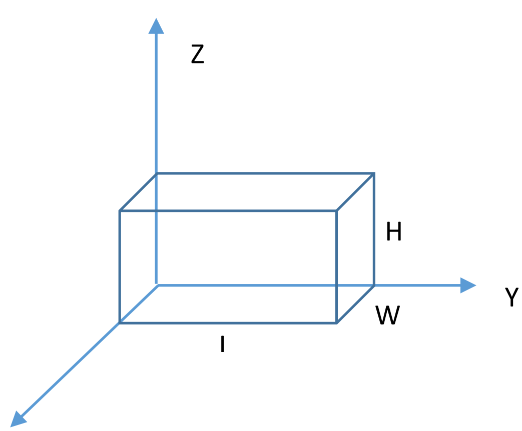
$$ I_{xx} = \iiint_V (y^2 + z^2)\rho d\upsilon \quad I_{yy} = \iiint_V (x^2 + z^2)\rho d\upsilon \quad I_{zz} = \iiint_V (x^2 + y^2)\rho d\upsilon $$

$$ I_{xy} = \iiint_V xy\rho d\upsilon \quad I_{xz} = \iiint_V xz\rho d\upsilon \quad I_{yz} = \iiint_V yz\rho d\upsilon $$

Example 6.1

$$ I_{xx} = \int_0^h \int_0^l \int_0^w (y^2 + z^2)\rho dxdydz = \frac{m}{3}(l^2 + h^2) $$

$$ I_{xy} = \int_0^h \int_0^l xy\rho dxdydz = \frac{m}{4} wl $$



Parallel Axis Theorem:

Inertial tensor of a mass in frame {A} w.r.t. frame {C} with its origin at the center of the mass.

$$^{A}I_{zz} = {}^{C}I_{zz} + m(x_c{}^2 + y_c{}^2)$$

$$^{A}I_{xy} = {}^{C}I_{xy} - m x_c y_c$$

$^{A}P_c = \begin{bmatrix} x_c & y_c & z_c \end{bmatrix}^T$ - Location of the center of mass in {A}.

Example 6.2

The frame {A} has its origin at $^{A}P_c = \frac{1}{2}\begin{bmatrix} w & l & h \end{bmatrix}^T$

$$^{C}I_{zz} = \frac{m}{12}(w^2 + l^2) \qquad {}^{C}I_{xy} = 0$$

***Newton's Equation on Force*:** $F = m\dot{v}_C$ at the center of mass

***Euler's Equation on Moment*:** $N = {}^{C}I\dot{\omega} + \omega \times {}^{C}I\omega$ at the center of mass

^{C}I = inertia tensor in frame {C} with its origin at the mass center

Newton-Euler Dynamic Equations

Derivation of angular acceleration

Forward angular velocity propagation

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\, {}^{i}\omega_i + \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1} \tag{5.45}$$

$$^{A}\dot{\Omega}_C = {}^{A}\dot{\Omega}_B + {}^{A}_{B}R\, {}^{B}\dot{\Omega}_C + {}^{A}\Omega_B \times {}^{A}_{B}R\, {}^{B}\Omega_C \tag{6.15}$$

Follow the derivation of (6.32) from (6.15) and (5.45)

Rewriting {C} with {i+1} and from (5.45)

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\, {}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\, {}^{i}\omega_i \times \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1} \tag{6.32}$$

For prismatic joints

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\, {}^{i}\dot{\omega}_i$$

Derivation of linear acceleration

From (6.12) and following similar steps taken for angular acceleration,

$$^{i+1}\dot{\upsilon}_{i+1} = {}^{i+1}_{i}R[{}^{i}\dot{\omega}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega_{i} \times ({}^{i}\omega_{i} \times {}^{i}P_{i+1}) + {}^{i}\dot{\upsilon}_{i}] \tag{6.34}$$

For *prismatic joints*, add two more terms to (6.34) per (6.10)

$$^{i+1}\dot{\upsilon}_{i+1} = {}^{i+1}_{i}R[{}^{i}\dot{\omega}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega_{i} \times ({}^{i}\omega_{i} \times {}^{i}P_{i+1}) + {}^{i}\dot{\upsilon}_{i}] + 2\,{}^{i+1}\omega_{i+1} \times \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \tag{6.35}$$

Linear acceleration of the center of mass, from (6.12)

Trace the steps taken in applying (6.12),

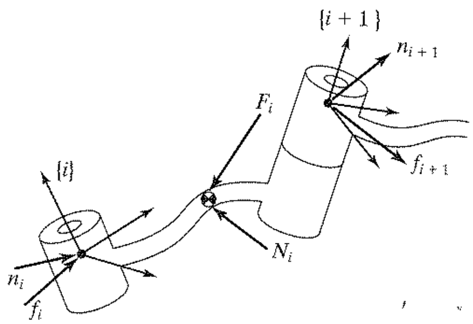
$$^{i}\dot{\upsilon}_{Ci} = {}^{i}\dot{\omega}_{i} \times {}^{i}P_{Ci} + {}^{i}\omega_{i} \times ({}^{i}\omega_{i} \times {}^{i}P_{Ci}) + {}^{i}\dot{v}_{i} \tag{6.36}$$

The inertial force and torque acting at the center of the mass:

From (6.32) and (6.36)

$$\begin{aligned} F_i &= m\dot{\upsilon}_{C_i} \\ N_i &= {}^{C_i}I\dot{\omega}_i + \omega_i \times {}^{C_i}I\omega_i \end{aligned} \tag{6.37}$$

H7Backward Iteration for Joint Forces and Torque



Force and torque balance equations at the center of mass of link i:

$$ {}^{i}F_i = {}^{i}f_i - {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} \tag{6.38}$$

$$ {}^{i}N_i = {}^{i}n_i - {}^{i}n_{i+1} + ({}^{i}P_i - {}^{i}P_{Ci}) \times {}^{i}f_i - ({}^{i}P_{i+1} - {}^{i}P_{Ci}) \times {}^{i}f_{i+1} \tag{6.39}$$

$$ {}^{i}P_i = 0 $$

Rearranging the equations and adding rotations;

$$ {}^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i \tag{6.41}$$

Figure out how this equation is related to (6.39).

$$ {}^{i}n_i = {}^{i}N_i + {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{Ci} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} \tag{6.42}$$

Finally, the joint torque is the Z component of the vector representing the inertial torque:

$$ \tau_i = {}^{i}n_i^T\,{}^{i}\hat{Z}_i \tag{6.43}$$

For prismatic joints:

$$ \tau_i = {}^{i}f_i^T\,{}^{i}\hat{Z}_i \tag{6.44}$$

Forward and backward iterations: Eq (6.45)-(6.53)

Forward - Link velocities and accelerations via the Newton-Euler (6.31)-(6.37).

Backward - Find joint forces and torques via (6.38)-(6.44).

See the Example in Section 6.7 – Simplified two link robot arm.

Check the vector cross multiplications at several places in the solution.

Dynamic Equations

State Space equation:

$$ \tau = M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta) \tag{6.59}$$

where,

$M(\Theta)$ = n x n mass matrix of terms containing $\ddot{\theta}_i, i = 1..n$

$V(\Theta,\dot{\Theta})$= n x 1 vector of centrifugal and Coriolis terms containing $\dot{\theta}_i, i = 1..n$

$G(\Theta)$= n x 1 vector containing a "g" gravity term.

Configuration Space equation:

$$\tau = M(\Theta)\ddot{\Theta} + B(\Theta)\left[\dot{\Theta}\dot{\Theta}\right] + C(\Theta)\left[\dot{\Theta}^2\right] + G(\Theta) \quad (6.63)$$

where,

$B_x(\Theta)$ = matrix of Coriolis coefficients

$\left[\dot{\Theta}\dot{\Theta}\right] = \left[\dot{\theta}_1\dot{\theta}_2 \quad \dot{\theta}_1\dot{\theta}_3 \quad \dots \quad \dot{\theta}_{n-1}\dot{\theta}_n\right]$, vector of joint velocity products (6.64)

$\left[\dot{\Theta}^2\right] = \left[\dot{\theta}_1^{\,2} \quad \dot{\theta}_1^{\,2} \quad \dots \quad \dot{\theta}_n^{\,2}\right]$, matrix of centrifugal coefficients (6.65)

Lagrangian Dynamic Formulationu

Quadratic form of manipulator kinetic energy, analogous to k = ½ mv^2

$$k(\Theta,\ddot{\Theta}) = \frac{1}{2}\dot{\Theta}^T M(\Theta)\dot{\Theta} \quad (6.71)$$

Potential energy: $u_i = -m_i{}^0 g T^o P_{Ci} + u_{ref\,i}$ (6.73)

$$u = \sum u_i$$

Lagrangian The difference between the kinetic energy and the potential energy

$$L(\Theta,\dot{\Theta}) = k(\Theta,\dot{\Theta}) - u(\Theta) \quad (6.75)$$

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta) \quad (6.59)$$

For Cartesian space,

$$F = M_x(\Theta)\ddot{X} + V_x(\Theta,\dot{\Theta}) + G_x(\Theta) \quad (6.91)$$

From $\tau = J^T(\Theta)F \rightarrow F = J^{-T}(\Theta)\tau$

$$F = J^{-T}[M_x(\Theta)\ddot{\Theta} + V_x(\Theta,\dot{\Theta}) + G_x(\Theta)] \quad (6.94)$$

From $\dot{X} = J\dot{\Theta} \quad \rightarrow \ddot{X} = \dot{J}\dot{\Theta} + J\ddot{\Theta} \quad \rightarrow \ddot{\Theta} = J^{-1}\ddot{X} - J^{-1}\dot{J}\dot{\Theta}$ (6.97)

Substituting (6.97) into (6.94)

$$F = J^{-T}[M_x(\Theta)(J^{-1}\ddot{X} - J^{-1}\dot{J}\dot{\Theta}) + V_x(\Theta,\dot{\Theta}) + G_x(\Theta)] \quad (6.98)$$

Then, $M_x(\Theta) = J^{-T}M(\Theta)J^{-1}$

$$V_x(\Theta,\dot{\Theta}) = J^{-T}[V(\Theta,\dot{\Theta}) - M(\Theta)J^{-1}\dot{J}\dot{\Theta}] \quad (6.99)$$

$$G_x(\Theta) = J^{-T}G(\Theta)$$

Cartesian configuration space torque

$$\tau = J^T[M_x(\Theta)\ddot{X} + V_x(\Theta,\dot{\Theta}) + G_x(\Theta)] \quad (6.104)$$

$$\tau = J^T M(\Theta)\ddot{X} + B_x(\Theta)\left[\dot{\Theta}\dot{\Theta}\right] + C_x(\Theta)\left[\dot{\Theta}^2\right] + G_x(\Theta) \tag{6.105}$$

where,

$B_x(\Theta)$ = matrix of Coriolis coefficients

$C_x(\Theta)$ = matrix of centrifugal coefficients

$\left[\dot{\Theta}\dot{\Theta}\right] = \left[\dot{\theta}_1\dot{\theta}_2 \quad \dot{\theta}_1\dot{\theta}_3 \quad \ldots \quad \dot{\theta}_{n-1}\dot{\theta}_n\right]$, a vector of the joint velocity products

$\left[\dot{\Theta}^2\right] = \left[\dot{\theta}_1^{\,2} \quad \dot{\theta}_1^{\,2} \quad \ldots \quad \dot{\theta}_n^{\,2}\right]$, a matrix of the centrifugal coefficients

Friction

Friction force $F\left(\Theta, \dot{\Theta}\right)$ may be added to (6.59) or (6.104) to account for the effect of friction on

Simulation

Numerical integration method is used to solve the acceleration problem of the manipulator.

$$\ddot{\Theta} = M^{-1}(\Theta)\left[\tau - V\left(\Theta, \dot{\Theta}\right) - G(\Theta) - F\left(\Theta, \dot{\Theta}\right)\right] \tag{6.115}$$

$\dot{\Theta}(t + \Delta t) = \dot{\Theta}(t) + \ddot{\Theta}(\Delta t)$, and

$$\Theta(t + \Delta t) = \Theta(t) + \dot{\Theta}(t)\Delta t + \frac{1}{2}\ddot{\Theta}(\Delta t)\Delta t^2 \tag{6.117}$$